MULTIPLICATIVE MAPS FROM $H\mathbf{Z}$ TO A RING SPECTRUM R - A NAIVE VERSION

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0. Introduction.

For a commutative ring B Stefan Schwede described in [S] the surprising connection between stable homotopy theory of commutative B-algebras and formal group laws over B. The stable homotopy operations of commutative simplicial B-algebras are described by the algebra $\pi_* DB$, where DB is a certain "classical" spectrum studied by Bousfield, Dwyer and others, see for example [D]. Schwede was able to describe the weak homotopy type of the space of multiplicative maps from the Eilenberg-MacLane spectrum $H\mathbf{Z}$ to DB in terms of formal group laws over B and their isomorphisms. But his methods seem to be much more general and should work in other situations as well. On the other hand the formal group laws over B are present in the description of DB so, in order to generalize his results one should start from defining something like "formal group law" even with the lack of formal power series.

In the present note we offer a definition of a formal group law in a ring spectrum R. With it we recover the weak version of the π_0-result of Schwede with any ring spectrum R at the place of DB. The obvious generalization of the full Schwede's result is clearly visible but we don't have any evidence to call it even "conjecture". At present we do not see methods of attacking this problem in full generality.

Every multiplicative map $H\mathbf{Z} \to R$ give R the structure of a ring over $H\mathbf{Z}$. We hope that observations presented in this note can be fruitful for our better understanding of the category of such objects.

We use here the language of Lydakis from [L], which we summarize in Section 1. The ring spectrum means here a Γ-ring in the sense of [L]. It means that a ring spectrum is a functor from the category of finite sets to simplicial sets with the extra structure. Maps between such objects are given in terms of natural transformations of functors. The word "naive" in our title refers to the fact that we work mostly with combinatorial structures only, so we don't have to use (up the last two pages) models for our spectra which give the correct homotopy type of the mapping space (fibrant - cofibrant replacement). We would like to thank the referee for many useful suggestions which helped us to improve the presentation. Especially the names used in the definitions 2.1 and 3.1 should be blamed on her/him.

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1. Preliminaries on Γ-spaces and Γ-rings.

Let $[n]$ denote the pointed set $\{0, 1, ..., n\}$ with 0 as a basepoint for a nonnegative integer n.

We want to distinguish three types of pointed set maps which will play the crucial role in the future. Two of them map $[n] \to [n-1]$ and the third one goes the other way around. The map $p^n_i : [n] \to [n-1]$ defined by $p^n_i(j) = j$ for $j < i$, $p^n_i(i) = 0$, $p^n_i(j) = j-1$ for $j > i$ will be called the ith restriction. For any $i < j \leq n$ we have the "summing" map $s^n_{i,j,k} : [n] \to [n-1]$ defined via the formula $s^n_{i,j,k}(i) = s^n_{i,j,k}(j) = k$, and the other elements $a \in [n]$ are mapped bijectively onto $[n-1] \setminus k$, preserving ordering. The third map $d^n_j : [n-1] \to [n]$ is injective, misses $j \in [n]$ and preserves the order.

The category Γ^{op} is a full subcategory of the category of pointed sets, with objects all $[n]$. The category of Γ-spaces is the full subcategory of the category of functors from Γ^{op} to pointed simplicial sets with objects satisfying $F[0] = [0]$ and maps given by the natural transformations of functors. Perhaps one should explain here that the notation Γ^{op} comes from the fact that our category is dual to Segal's category Γ from [Se]. Every Γ-space can be prolonged by direct limits to the functor defined on the category of pointed sets. In our notation we will not distinguish between the Γ-space and the described above extension. We will use capital letters K, L, ... for denoting pointed sets. In the future, if we need ordering of the pointed set $[n] \wedge [m]$ which identifies it with $[nm]$ we will use always the inverse lexicographical order.

Convention: If it causes no misunderstandings having pointed sets K and L and a pointed map $f : K \to L$ we write f instead of $F(f)$ for the induced map $F(K) \to F(L)$.

For a Γ-space F let RF denote the $\Gamma \times \Gamma$ space defined as $RF(K, L) = F(K \wedge L)$. Having two Γ-spaces F and F' we can form their exterior smash product $\Gamma \times \Gamma$-space $F \tilde{\wedge} F'$ which is defined by the formula $F \tilde{\wedge} F'(K, L) = F(K) \wedge F'(L)$. Then the smash product of F and F' is the universal Γ-space F'' with a map of $\Gamma \times \Gamma$-spaces $F \tilde{\wedge} F' \to RF''$, (see [L, Remark 2.4]). Moreover, if we denote by $\mathcal{GS}$ the category of Γ-spaces and by $\mathcal{GSS}$ the category of $\Gamma \times \Gamma$-spaces then for given Γ-spaces F_1, F_2 and F_3 we have (following [L,Theorem 2.2]):

$$\mathcal{GS}(F_1 \wedge F_2, F_3) = \mathcal{GSS}(F_1 \tilde{\wedge} F_2, RF_3)$$

Remark 1.1: The symmetric group Σ_n acts on the set $\{0, 1, ..., n\}$ by permuting $\{1, ..., n\}$ and hence this group acts on $F[n]$ for any Γ-space F. We will use this action restricted to various subgroups of Σ_n in the future.

Let $\mathbf{S}$ denote the Γ-space defined by the identity functor. We say that Γ-space F is a Γ-ring if there are maps $\eta : \mathbf{S} \to F$ called the unit and $\mu : F \wedge F \to F$ called the multiplication satisfying usual associativity and unit conditions (see [L, 2.13]).

Remark 1.2: By our previous observations μ is determined by a map $\tilde{\mu} : F \tilde{\wedge} F \to RF$, which is fully determined by a collection of maps $\tilde{\mu} : F[n] \wedge F[m] \to F([n] \wedge [m])$ natural in $[n]$ and $[m]$ and satisfying obvious associativity conditions.

Let us introduce one more piece of notation. We will say that a Γ-ring R is discrete if for any pointed set K , $R(K)$ is just a set considered as a simplicial set in a trivial way. Assume that R is a discrete Γ-ring. Then $R[1]$ is a unital monoid with zero. Moreover η takes $1 \in \mathbf{S}[1]$ to the unit of $R[1]$.

Remark 1.3: Assume that R is a discrete Γ-ring. Then the map $\tilde{\mu} : R(K) \wedge R(L) \to R(K \wedge L)$ of 1.2 is a map of sets which is associative with respect to the smash product of pointed sets. It means that if $p \in R(K)$ and $q \in R(L)$ then it makes sense to say that the product of p and q belongs to $R(K \wedge L)$ which, of course , means that $\tilde{\mu}(p, q) \in R(K \wedge L)$. We will usually write the result of such multiplication as $pq \in R(K \wedge L)$.

2. Multiplicative maps from $H\mathbf{N}$ to a discrete Γ-ring R

We want to study multiplicative maps from the Eilenberg-MacLane spectrum $H\mathbf{Z}$ to a Γ-ring R. But the answer is a bit technical and we will postpone it until section 3. In the present section we will study maps from the spectrum stably equivalent to $H\mathbf{Z}$ which is easier to study. Let us start from recalling a Γ-ring model of $H\mathbf{Z}$. As a functor $H\mathbf{Z}$ takes K to a reduced free abelian group generated by K. The map

$$\eta : \mathbf{S} \to H\mathbf{Z}$$

is given by the embedding of generators. The multiplication map

$$\mu : \tilde{\mathbf{Z}}(K) \wedge \tilde{\mathbf{Z}}(L) \to \tilde{\mathbf{Z}}(K \wedge L)$$

is defined via the formula

$$(\sum_{k \in K} a_k k) \wedge (\sum_{l \in L} b_l l) \mapsto \sum_{k \wedge l \in K \wedge L} a_k b_l (k \wedge l)$$

The Γ-ring $H\mathbf{N}$ is defined with the same formulas but with the additive monoid of natural numbers (with 0) instead of integers. The embedding $H\mathbf{N} \to H\mathbf{Z}$ induces a stable equivalence of spectra because for any $k > 0$ the map $H\mathbf{N}(S^k) \to H\mathbf{Z}(S^k)$ is a homotopy equivalence by a theorem of Spanier [Sp, Theorem 4.4]. This map is obviously multiplicative but there is no nontrivial multiplicative map going the other way.

It is easy to realize that Schwede's map $H\mathbf{Z} \to DB$ associated to a formal group F is determined uniquely by saying that the image of $(1, 1) \in H\mathbf{Z}[2]$ is F. We comment on this more later but now we should define the possible images of $(1, 1) \in H\mathbf{N}[2]$ in the case of an arbitrary Γ-ring R. Below we write the first definition of a formal group law in a Γ-ring R.

Definition 2.1: A formal sum law in a Γ-ring R is an element $w \in R[2]$ which satisfies the following properties:

1. $p_1^2(w) = 1$, $p_2^2(w) = 1$,
2. any power $w^k \in R[2^k]$ is fixed under the action of the symmetric group Σ_{2^k}.

Theorem 2.2: Let R be a discrete Γ-ring. Then every formal sum law in R determines the multiplicative map $\phi : H\mathbf{N} \to R$.

Proof. Let w be a formal sum law in R. Let $1_n = (1, ..., 1) \in H\mathbf{N}[n]$. We want to show that associating $1_{2^n} \mapsto w^n$ defines the desired map ϕ.

Observe first that any element $(n_1, ..., n_k) \in \mathbf{N}[k]$ can be presented as an image of 1_n for a certain n . So our map is uniquely determined on elements 1_n: if for a pointed map $f : [n] \to [k]$ we have $f(1_n) = (n_1, ..., n_k)$ then we must have

$$\phi(n_1, ..., n_k) = f(w^n)$$

Hence we only have to show that ϕ is well defined by the formula above.

First of all, for the given $k$tuple $(n_1, ..., n_k) \in \mathbf{N}[k]$ there is a minimal n such that 1_{2^n} maps to $(n_1, ..., n_k)$. Obviously n is equal to the minimal natural number n which satisfies the condition $2^n \geq \sum_{i=1}^{k} n_i$. Of course there are many ways of mapping 1_{2^n} to $(n_1, ..., n_k)$ but all of them give the same definition of $\phi(n_1, ..., n_k)$ because of the condition 2 of the definition of the formal sum law.

Assume now that $g(1_{2^m}) = (n_1, ..., n_k)$ for a certain map g with $n < m$. Then it is easy to see that g factors through 1_{2^n}. Hence our proof that ϕ is well defined will be finished if we show:

Lemma 2.2.1: For any k, w^{2^k} maps to $w^{2^{k-1}}$ under any map f which satisfies $f(1_{2^k}) = 1_{2^{k-1}}$.

Proof of 2.2.1. Assume first that f takes last 2^{k-1} coordinates to zero. In other words, using the fact that

$$[2^k] = [2^{k-1}] \wedge [2]$$

we can write

$$f = Id_{[2^{k-1}]} \wedge p_2^2$$

Then

$$f(w^{2^k}) = f(w^{2^{k-1}} \cdot w) = (Id_{[2^{k-1}]} \wedge p_2^2)(w^{2^{k-1}} \cdot w) = w^{2^{k-1}} \cdot 1 = w^{2^{k-1}}$$

Now observe that, because of the property 2 of formal sum laws, it is enough to consider only maps like f above. Any other f' which takes 1_{2^k} to $1_{2^{k-1}}$ differs from f by an action of an element from Σ_{2^k}.

We will finish the proof of the theorem if we show that our map ϕ obtained in the way described above is multiplicative. To check this observe first that if $f(1_{2^n}) = (n_1, ..., n_k)$ and $g(1_{2^m}) = (m_1, ..., m_l)$ then $(f \wedge g)(1_{2^n} \cdot 1_{2^m}) = (f \wedge g)(1_{2^{n+m}}) = (n_1, ..., n_k)(m_1, ..., m_l)$ as elements of $H\mathbf{N}[kl]$. We calculate further:

$$\phi((n_1, ..., n_k) \cdot (m_1, ..., m_l)) = \phi(f \wedge g(1_{2^{n+m}})) = f \wedge g\phi(1_{2^{n+m}}) = f \wedge g(w^n \cdot w^m) =$$

$$= f(w^n) \cdot g(w^m) = \phi(f(1_{2^n})) \cdot \phi(g(1_{2^m})) = \phi(n_1, ..., n_k) \cdot \phi(m_1, ..., m_l)$$

and the proof is finished.

We will come to the issue when two formal sum laws give homotopic maps later in a more general setting. But from the proof of theorem 2.2 it is easy to derive the following observation:

Remark 2.3: Observe that our assumption that R is discrete is not important. We could define a formal sum law in R as a 0-simplex of $R[2]$ and the rest would go through by the same arguments.

3. Multiplicative maps from $H\mathbf{Z}$ to a discrete Γ-ring R.

Now we move towards studying multiplicative maps $H\mathbf{Z} \to R$. We would like to define the formal group laws in this situation in such a way that we get the same statement as in 2.2. But first of all let us identify the complications which occur when we allow negative coordinates in our source Γ-ring. The problem is that while working with an arbitrary Γ-ring R one does not have any natural way of defining maps coming from multiplying one "variable" by -1. That was not a problem in the case of DB. More generally this is not a problem in the case of any Γ-ring coming from the composition of functors

$$T \circ L : \Gamma^{op} \to Sets_*$$

where L is the linearization functor from sets to the category B_{free} of free modules over some ring B and $T : B_{free} \to Sets_*$. We plan to study such situations in the forthcoming paper but now we would like to define the formal group law in full generality overcoming the difficulty described above.

But before the definition we have to describe a particular type of an action of $\Sigma_{2^{k-1}} \times \Sigma_{2^{k-1}}$ on $F[2^k]$ for any Γ-space F. This action will ba called *special* later on. Let for any k, $\pm 1_k$ be equal to $(1, -1)^k \in H\mathbf{Z}[2^k]$. Our convention on ordering smash products of pointed sets permits to split $[2^k] = A_+ \vee A_-$ accordingly to the rule that $(1, -1)^k$ has 1 at the coordinates from A_+ and -1 at A_-. In a more formal way we can say that an element $i \in \{1, ..., 2^k\}$ belongs to A_+ it and only if the binary expansion of $i-1$ has an even number of digits "1". There is also a "coordinate-free" way of describing the splitting. If we identify the set $[2^k]$ with the set of subsets of the set $\{1, 2, ..., k\}$ then A_+ (A_-) consists of sets of even (odd) order.

The special action of $\Sigma_{2^{k-1}} \times \Sigma_{2^{k-1}}$ on $F[2^k]$ is defined as follows: if $a \times b \in \Sigma_{2^{k-1}} \times \Sigma_{2^{k-1}}$ then a permutes the coordinates from A_+ and b permutes the rest of the coordinates. Let σ be the nontrivial element of Σ_2.

Definition 3.1: A formal difference law in a Γ-ring R is an element $r \in R[2]$ satisfying the following properties:

1. $p^2_2(r) = 1$, $s^2_{1,2,1}(r) = 0$
2. $p^2_1(r)r = rp^2_1(r) = \sigma(r)$ in $R[2]$
3. any power $r^k \in R[2^k]$ is fixed under the special action of $\Sigma_{2^{k-1}} \times \Sigma_{2^{k-1}}$.

4. for any k , $i < j$ and l such that $i \in A_+$ and $j \in A_-$ or $j \in A_+$ and $i \in A_-$ we have

$$s^{2^k}_{i,j,l}(r^k) = d_l^{2^k-1} p_i^{2^k-1} p_j^{2^k}(r^k)$$

Observe first that $p_1^2(r)$ plays a role of -1 in $R[1]$ because we can calculate;

$$(p_1^2(r))^2 = (p_1^2 \wedge p_1^2)(r^2) = (p_1^2 \wedge p_1^2) \circ \tau)(r^2) = (p_2^2 \wedge p_2^2)(r^2) = (p_2^2(r))^2 = 1$$

where τ is a special permutation in $\Sigma_2 \times \Sigma_2$ given by the transposition $(1,4)$. We can now compare our new definition with results and definitions from Section 2. We check that if r is a formal difference law in a Γ-ring R then $w = p_2^3 \circ p_2^4(r^2) \in R[2]$ is a formal sum law in the sens of definition 2.1. Indeed:

$$p_2^2(w) = p_2^2 \wedge p_2^2(r^2) = (p_2^2(r))^2 = 1$$

Similarly:

$$p_1^2(w) = p_1^2 \wedge p_1^2(r^2) = (p_1^2(r))^2 = 1$$

Moreover

$$\sigma(w)) = p_2^3 \circ p_2^4(\tau(r^2)) = p_2^3 \circ p_2^4(r^2)$$

and hence w is fixed under the action of Σ_2. Let p denote $p_2^3 \circ p_2^4$. By naturality of the smash product and multiplication maps we have the following commutative diagram:

$$\begin{array}{ccc} R[4] \wedge R[4] & \longrightarrow & R[16] \\ \downarrow & & \downarrow \\ R[2] \wedge R[2] & \longrightarrow & R[4] \end{array}$$

where the left vertical arrow is given by $p \wedge p$ and horizontally we have multiplication maps. Then the right vertical map is defined by the set map which takes 4 elements of A_+ bijectively to nonzero elements of $[4]$ and the rest elements of $[16]$ to 0 . Hence the action of any permutation from Σ_4 on $w \wedge w$ rises to the special permutation acting on $R[16]$. This argument generalizes easily to higher degrees because $p^{\wedge k}$ maps bijectively 2^k elements of $A_+ \subset [4^k]$ to nonzero elements of $[2^k]$ and has value 0 otherwise.

Thinking about our definition as if it was a definition of a formal group law in an ordinary sense we can give an interpretation of the most of the structure described in 3.1. An element $p_1^2(r)$ plays a role of -1 in the "commutative group structure" defined by r. Hence it commutes "with other elements". Condition 1 is always included in the general definition of a formal group law. The same can be said about condition 3 - in the classical case of formal power series this kind of invariance property is indirectly in the definition of a formal group law.

The condition 4 is new and makes the situation technically more complicated. It is hard to imagine what would be the abstract meaning of it. This condition is strongly related to the fact that $1 + (-1) = 0$ in $\mathbf{Z}$ which is a very additive condition, having no meaning in the structure of an arbitrary Γ-ring . The simplest explanation which one can imagine for the need of 4 is the following remark: condition 4 is an extension of the second formula from condition 1 to higher degrees, which is needed when we face the lack of additivity. The good news is that condition 4 from 3.1 is often satisfied in interesting cases, namely in Γ-rings coming from algebraic theories. This is the case of the Γ-ring DB. We are not going to define here what is an algebraic theory and what is the definition of a Γ-ring associated to it. Instead we send the interested reader to [S2, Section 2].

Remark 3.2: Let T^s be a Γ-ring associated to the algebraic theory T. Then condition 4 of 3.1 is satisfied for T^s as a consequence of condition 1.

Proof (the sketch): We will follow [S1, Section 2] without further explanations. Observe first that formula 4 of 3.1 for $k = 1$ is equivalent to the second equality of the condition 1 of 3.1 and hence is satisfied. By definition $T^s[n] = hom_T([n], [1])$ and the multiplication

$$T^s[n] \wedge T^s[m] \to T^s[nm]$$

is obtained from composition. It means that we can write it as

$$\alpha \wedge \beta \mapsto \beta \circ (\alpha, ..., \alpha)$$

with our convention of identifying $[n] \wedge [m]$ with $[nm]$. So in the notation as above $r^k = r^{k-1} \circ (r, ..., r)$ and the value of the map $s^{2^k}_{i,j,l}$ on r^k is the same as if we apply $s^2_{1,2,1}$ to one of the r's in the bracket by naturality and property 3 of 3.1. So

$$s^{2^k}_{i,j,l}(r^k) = r^{k-1} \circ (r, ..., r, 0, r, ..., r) = d_l^{2^k-1} p_i^{2^k-1} p_j^{2^k}(r^k)$$

Definition 3.3: A *homomorphism* $a : r_1 \to r_2$ of formal difference laws in R is an element $a \in R[1]$ satisfying $ar_1 = r_2 a$. An invertible homomorphism is called an *isomorphism* of formal difference laws. An isomorphism is called *strict* if it maps to the unit component of R under the map $R[1] \to \pi_0 R$.

Perhaps for completeness it is worth here to recall the definition of the map $R[1] \to \pi_0 R$. According to [S2, lemma 1.2] $\pi_0 R$ can be presented as the cokernel of the map

$$\tilde{Z}p_2^2 + \tilde{Z}p_1^2 - \tilde{Z}s^2_{1,2,1} : \tilde{Z}[R[2]] \to \tilde{Z}[R[1]]$$

Then our map can be described as an embedding of generators composed with the quotient map described above.

Theorem 3.4: Let R be a discrete Γ-ring. Then every formal difference law in R determines the multiplicative map $\phi : H\mathbf{Z} \to R$.

Proof. Let r be a formal difference law in R. We want to show that associating $\pm 1_n \mapsto r^n$ defines the desired map.

Observe first that any element $(n_1, ..., n_k) \in \mathbf{Z}[k]$ can be presented as an image of $\pm 1_n$ for certain n. Hence our map is uniquely determined on elements $\pm 1_n$ and we only have to show that ϕ is well defined.

We would like to follow the proof of 2.2 but the situation is different now. The proof of 2.2 was based on the fact that an element $(n_1, ..., n_k) \in \mathbf{N}[k]$ was equal to the image of 1_n in a unique way up to a permutation which acted trivially on the corresponding power of r. This is now not the case: 1 and -1 from different coordinates in $\mathbf{Z}[\mathbf{k}]$ can cancel either by mapping coordinates to the base point or by the summing map. In the case of the proof of 2.2 we had only to consider the first possibility.

First of all, as previously, for the given $k$tuple $(n_1, ..., n_k) \in \mathbf{Z}[k]$ there is minimal n such that $\pm 1_n$ maps to $(n_1, ..., n_k)$ by the map f'. We can assume that all n_is are different from 0. There is a special permutation σ such that $f = f' \circ \sigma$ takes first $|n_1|$-coordinates in $\sigma((\pm 1)^k)$ with the same sign as n_1 to n_1, next $|n_2|$ coordinates with correct signs to n_2 and so on. Let $N_k = \Sigma_{i=1}^k |n_i|$. It means that all ones and minus ones on the other $n - N_k$ coordinates have to cancel to zero. Assume that $a < b$ and we have 1 on ath coordinate and -1 on bth and they cancel each other (add to 0). Then obviously $f = f \circ d_b^{2^n} \circ s_{a,b,a}^{2^n}$ as the maps of pointed sets and we can iterate this process composing with more pairs of maps $d_*^{2^n} \circ s_{*,*,*}^{2^n}$. But observe that, because of condition 4 of the definition of the formal difference law we have

$$f(r^n) = f \circ d_b^{2^n} \circ s_{a,b,a}^{2^n}(r^n) = f \circ d_b^{2^n} \circ d_a^{2^n-1} \circ p_a^{2^n-1} \circ p_b^{2^n}(r^n)$$

It means that the value of f on r^n is the same as the value of a map which takes a and b to a base point. Iterating this process we see that we have justified the lemma:

Lemma: 3.4.1: Let $g : [2^n] \to [k]$ be a map which agrees with f on the N_k elements chosen as it is described above and takes the rest to the base point. Then

$$f(r^n) = g(r^n)$$

Hence we see that our map ϕ is well defined. Whichever map f taking $\pm 1_n$ to $(n_1, ..., n_k)$ we use it will have the same value on r^n as the map g from 3.4.1. Checking that if $f(\pm 1_n) = (n_1, ..., n_k) = h(\pm 1_l)$ then two definition of $\phi(n_1, ..., n_k)$ agree goes essentially the same way as in the proof of 2.2 and is left to the reader. Similarly one can show the multiplicativity of ϕ.

Remark 3.5: Any multiplicative map $\phi : H\mathbf{Z} \to R$ determines a formal difference law in R. It is given by the formula $r = \phi(\pm 1_2)$. Hence we can say, that the set of formal difference laws in R is in natural bijection with the set of multiplicative maps $H\mathbf{Z} \to \mathbf{R}$.

Perhaps it is now a good point to present how our definition works in known cases, for example in the case of the spectrum DB. In Section 2 we mentioned that every formal sum law in DB determines the formal group law in the ordinary sense. Observe now that a formal difference law $r \in R[2]$ in the sense of definition 3.1 determines its sum version $w \in R[2]$ by the formula

$$w = p_2^3 \circ p_2^4(r^2).$$

Moreover the map $H\mathbf{N} \to R$ defined by w factors through the map $H\mathbf{Z} \to \mathbf{R}$ defined by r.

As another example we would like to show how our definition works in the case of endomorphism Γ-ring. This notion is probably less known so we sketch the definition of it following the presentation from [S, 13.3].

Example: Let $\mathcal{C}$ be a category with 0-object and finite coproducts. The natural enrichment of $\mathcal{C}$ over Γ^{op} is given by

$$X \wedge [k] \; = \; X \sqcup ... \sqcup X \quad (k - fold \;\; coproduct).$$

Every object X in $\mathcal{C}$ has its endomorphism Γ-ring denoted $End_{\mathcal{C}}(X)$ defined by

$$End_{\mathcal{C}}(X)([k]) = Hom_{\mathcal{C}}(X, X \wedge [k]).$$

The unit map $\mathbf{S} \to End_{\mathcal{C}}(X)$ comes from the identity map in $End_{\mathcal{C}}(X)([1])$. The multiplication is induced by the composition product

$$End_{\mathcal{C}}([k])(X) \wedge End_{\mathcal{C}}(X)([l]) \to End_{\mathcal{C}}(X)([k] \wedge [l])$$

$$f \wedge g \mapsto (f \wedge [l]) \circ g.$$

As Schwede points out, every abelian cogroup object structure on X determines the map $H\mathbf{Z} \to End_{\mathcal{C}}(X)$ defined as follows. At a finite pointed set $[k]$ the map

$$H\mathbf{Z}([k]) = \tilde{\mathbf{Z}}[k] \to Hom_{\mathcal{C}}(X, X \wedge [k])$$

is an additive extension of the map which sends $i \in [k]$ to the ith coproduct inclusion $X \to X \sqcup ... \sqcup X$.

Observe now, that every formal difference law $r \in End_{\mathcal{C}}(X)[2] = Hom_{\mathcal{C}}(X, X \sqcup X)$ defines the abelian cogroup structure on X. The co-addition is given by a sum version of r

$$p_2^3 \circ p_2^4(r^2) \in Hom_{\mathcal{C}}(X, X \sqcup X).$$

It is abelian because of the invariance of formal sum laws under the action by permutations. By the same reason the associativity condition is fulfilled. The co-inverse is given by $p_1^2(r) \in End_{\mathcal{C}}(X)([1]) = Hom_{\mathcal{C}}(X, X)$. The co-unit equals $s^2_{1,2,1}(r) \in End_{\mathcal{C}}(X)([1]) = Hom_{\mathcal{C}}(X, X)$. The described by Schwede (and recalled above) map $H\mathbf{Z} \to End_{\mathcal{C}}(X)$ agrees with the one obtained by theorem 3.4 from r.

We suggest the reader to work out by himself, how our theory works in the case of matrix Γ-rings, see [S, 13.5]. Below we come back to the question when two formal difference laws define homotopic maps $H\mathbf{Z} \to R$.

Theorem 3.6. Two strictly isomorphic formal difference laws determine homotopic maps in the space of maps $H\mathbf{Z} \to R$.

Recall that when $a \in R[1]$ then multiplication by a from the left or from the right determines the map $m_a : R \to R$. Because left and right multiplications are formally the same we will assume that m_a comes from the multiplication from the left. The theorem 3.6 follows easily from the following lemma.

Lemma 3.7. Assume that a and b are two elements in $R[1]$ which determine the same element in $\pi_0(R)$ under the obvious map $R[1] \to R$. Then multiplication maps m_a and m_b are homotopic.

Proof. The statement of the lemma follows directly from the definitions, if one carefully investigates what does it mean that $a \in R[1]$ determines an element in $\pi_0(R)$. Observe that choosing $a \in R[1]$ we uniquely choose a map $f_a : \mathbf{S} \to R$: it is fully described on the set [1] where we put $\mathbf{S}[1] \ni 1 \mapsto a \in R[1]$. In higher degrees our map is determined by this data because every $i \in \mathbf{S}[n]$ can be viewed as the image of the map $[1] \to [n]$ taking 1 to i.

Moreover observe that on the set [1] our map can be described as the unit map η multiplied from the left by a. By naturality of the multiplication map we see that our map f_a is equal to η composed with the multiplication from the left by a. Observe now that we can decompose the map m_a as a composition

$$R \to \mathbf{S} \wedge R \to R \wedge R \to R$$

where the first map is an isomorphism, the second is given by $f_a \wedge id$ and the third is given by the multiplicative structure μ of the Γ-ring R.

Now we can come back to the proof of 3.7. By the assumption f_a and f_b determine the homotopic maps of spectra. From this we get that $f_a \wedge id$ is homotopic to $f_b \wedge id$ and hence m_a and m_b give us the homotopic maps of spectra.

Let come back to the proof of 3.6. We know that r_1 and r_2 are strict isomorphic, and the isomorphism is given by an invertible element $a \in R[1]$. Let ϕ_1 (ϕ_2) denote the map $H\mathbf{Z} \to \mathbf{R}$ determined by r_1 (r_2). Then

$$\phi_2 = m_{a^{-1}} \circ \phi_1 \circ m_a$$

By the assumption m_a and $m_{a^{-1}}$ are homotopic to the m_1, hence to the identity map. This finishes the proof of 3.6.

The referee suggested the following interesting generalization of the considerations above to the case when R is not discrete. In the latter case, for every natural n, the nth simplicial degree of $R[K]$ assemble to a discrete Γ-ring R_n. Hence, in the case of R not being discrete, we can talk about simplicial set $FDL(R)$ of formal difference laws in R which in degree n has the set of formal difference laws in R_n. Similarly we can consider the simplicial set $\Gamma(H\mathbf{Z}, R)$ of multiplicative maps from $H\mathbf{Z}$ to R which in degree n has the set of such maps to R_n. Then the theorem 3.5 can be stated as

Theorem 3.8: Simplicial sets $FDL(R)_*$ and $\Gamma(H\mathbf{Z}, \mathbf{R})_*$ are naturally isomorphic.

Moreover the story goes further taking into account the action of the invertible elements of $R[1]$. Let G_n be the group of invertible elements in $R_n[1]$. they assemble to a simplicial group G_* and this simplicial group acts on both simplicial sets from 3.8 by conjugation. With all this structure in mind we can generalize 3.6 to the following statement.

Theorem 3.9: The homotopy orbit sets $FDL(R)_{hG_*}$ and $\Gamma(H\mathbf{Z}, R)_{hG_*}$ are isomorphic.

Now we would like to comment a little on the homotopical meaning of our constructions. As one can see, the proof of 3.6 was derived directly from the definitions. But of course we would like to know whether two strictly isomorphic formal difference laws define homotopic maps in the space of multiplicative maps from $H\mathbf{Z} \to R$ or, equivalently, the same element in the 0th homotopy group of the space of multiplicative maps $H\mathbf{Z} \to R$ as in [S]. The answer here is not easy to achieve or even to formulate the conjecture. Our constructions depended heavily on the small model of $H\mathbf{Z}$ which is not cofibrant. Schwede's homotopical calculations were possible also because of the definition of DB-spectrum and its closed relations to symmetric algebra. With the lack of all these structures we can only propose the following weak homotopical statement:

Proposition 3.10: Let r_1 and r_2 be two strictly isomorphic formal difference laws in a Γ-ring R. There exists a weak equivalence of Γ-rings $h : R \to R_3$ such that maps defined by r_1 and r_2 composed with h are homotopic in the space of multiplicative maps $H\mathbf{Z} \to R_3$

Proof. We will be sketchy here because the proof is taken directly from [S]. For any Γ-ring R invertible elements in $R[1]$ which maps to the unit component of R form a group G which acts by conjugation on $R[2]$ and in general on R. Two formal group laws $r_1,\ r_2 \in R[2]$ are strictly isomorphic if they are in the same orbit of this action. Our problem would be solved if we could extend our conjugation action described above to the action of the whole unit component of R.

Following [S, section 3] we first choose R^f to be a stably fibrant replacement of R in a correct model category structure. Then we define homotopy units R^* as the union of invertible components of the simplicial monoid $R^f[1]$. We have $\pi_0 R^* = units(\pi_0 R)$ and $\pi_i R^* = \pi_i R$ for $i \geq 1$. The stable equivalence $R \to R^f$ gives us a homomorphism $\phi : G \to R^f[1]$ of simplicial monoids with the image in R^*. We want to extend the conjugation action of G to R^*. The problem is that the conjugation action uses strict inverses while R^* is only a group-like simplicial monoid. The construction how to get around this difficulty goes in several steps (see [S, section 4] for the details).

Step 1. We factor the map $G \to R^*$ into $G \to cR^* \to R^*$ in the correct model category structure of simplicial monoids where the first map is a cofibration and the second an acyclic fibration . Let UR^* denote the group completion of cR^*. Then UR^* is a simplicial group and Lemma 4.3 of [S] tells us that the map $cR^* \to UR^*$ is a weak equivalence.

Step 2. Let $\mathbf{S}[cR^*]$ be the monoid Γ-ring with coefficients in the sphere spectrum. We take the obvious map $\mathbf{S}[cR^*] \to R^f$ and factor it in the model category of Γ-rings as a cofibration followed by an acyclic fibration

$$\mathbf{S}[cR^*] \to R_1 \to R^f.$$

Then we define another Γ-ring R_2 as a pushout, in the category of Γ-rings of

$$\begin{array}{ccc} \mathbf{S}[cR^*] & \to & R_1 \\ \downarrow & & \downarrow \\ \mathbf{S}[UR^*] & \to & R_2 \end{array}$$

Lemma 4.4 of [S] tells us that the map $R_1 \to R_2$ is a stable equivalence.

Step 3. Now we define R_3 to be a stably fibrant replacement of R_2. The induced map $\mathbf{S}[UR^*] \to R_3$ induces a weak equivalence between UR^* and the invertible components of $R_3[1]$. The simplicial group UR^* acts by conjugation on R_3 via homomorphisms of Γ-rings and this action extends the action of G.

Final remark. Of course it is very tempting to speculate that the weak homotopy type of the full space of multiplicative maps $H\mathbf{Z} \to R$ should be described via the classifying space of the groupoid of formal difference laws and strict isomorphism, as it is proved in [S] in the case of the spectrum DB. So far we do not see any way of attacking this problem in full generality.

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