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Lifting functors from $\mathcal{F}$ to $\mathcal{P}$

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1. Introduction.

Let p be a prime number. Let $\mathcal{F}$ denote the category of functors from finite dimensional vector spaces over F_p to vector spaces over F_p. We write $deg(F)$ for the degree of $F \in \mathcal{F}$ in the sense of Eilenberg and Mac Lane. A functor $F \in \mathcal{F}$ is called *finite* if it takes finite dimensional values and $deg(F) < \infty$. Let $\iota : \mathcal{P} \to \mathcal{F}$ denote the forgetful functor from the category of strict polynomial functors in the sense of Suslin-Friedlander to the category $\mathcal{F}$. As usual $\mathcal{P}_d$ denotes the subcategory of $\mathcal{P}$ of functors of homogeneous degree d. We have a decomposition

$$\mathcal{P} \; = \; \bigoplus_d \mathcal{P}_d.$$

If $P \in \mathcal{P}_d$ then the number d is called the weight of P and it is denoted $w(P)$. Observe that an inequality

$$deg(\iota(P)) \leq w(P)$$

holds for any $P \in \mathcal{P}$, where for a nonhomogeneous P, $w(P)$ denotes the highest weight of its homogeneous pieces. We say that $F \in \mathcal{F}$ lifts to $\mathcal{P}$ if there exists $P \in \mathcal{P}$ such that $\iota(P) = F$. Our goal is to find some necessary and sufficient cohomological conditions which characterize these functors in $\mathcal{F}$ which can be lifted to $\mathcal{P}$. The definition of $\mathcal{P}$ and the inequality relating $w(P)$ and $deg(\iota(P))$ imply that we restrict our considerations to finite functors in $\mathcal{F}$.

The category $\mathcal{P}$ is more accessible for cohomological calculations than $\mathcal{F}$. Most of the known results on the *Ext*-calculations in $\mathcal{F}$ were achieved for functors which are in the image of ι by using the results which compare $Ext_{\mathcal{F}}(\iota(P), \iota(Q))$ and $Ext_{\mathcal{P}}(P, Q)$, see [FFSS] for the strongest results in this direction. Section 5 of the present paper contains two examples of cohomological problems in $\mathcal{F}$ which were important for the author by other reasons. The solution of them is presented but only for functors which can be lifted to $\mathcal{P}$. The general answer in $\mathcal{F}$ is still not known to the author.

The paper is organized as follows. Section 2 contains very basic observations, which show only than one should not expect the simple solution for the problem od lifting functors from $\mathcal{F}$ to $\mathcal{P}$. This section ends which two examples and one of them presents a functor which cannot be lifted. Looking for the solution to our main problem we want to use as a tool extension groups and *Ext*-algebras in our categories. It is known that they do not determine directly the structure of an abelian category. For this the stronger, A_∞ structure is needed. We present the important results on this structure in Section 3. In Section 4 we prove our main theorem (Theorem 4.3) which summarizes to the statement that a functor $F \in \mathcal{F}$ can be lifted to $\mathcal{P}$ iff certain classes in $Ext^1_{\mathcal{F}}(.,.)$ can be lifted. The content of Section 5 was described above.

2. Preliminary observations.

As usual, we write $F^{(1)}$ for the precomposition of a functor F in $\mathcal{F}$ or $\mathcal{P}$ with the Frobenius twist. This operation is equal to identity in $\mathcal{F}$ but in $\mathcal{P}$ rises the weight of a functor p times. We write $F^{(i)}$ for the i-fold precomposition with the Frobenius twist.

Lemma 2.1. Let $F \in \mathcal{F}$ be a simple functor. Then F lifts to $\mathcal{P}$ and all its lifts are of the form $P = F^{(i)}$ for some natural i.

Proof. The functor ι is exact and if $\iota(P) = 0$ then $P = 0$. Hence any lift of a simple object in $\mathcal{F}$ must be simple in $\mathcal{P}$. The result then follows from the Kuhn's description of simple objects in $\mathcal{F}$ and $\mathcal{P}$ as it is written in [K3, Section 7].

Let $l_{\mathcal{C}}(X)$ denote the length of the composition series of an object X in the category $\mathcal{C}$. We have the obvious lemma:

Lemma 2.2. For any $P \in \mathcal{P}$, $l_{\mathcal{P}}(P) \leq l_{\mathcal{F}}(\iota(P))$.

Proof. The composition series of P maps to the series of subobjects of F. The lemma follows then from the fact that ι takes proper inclusions to proper inclusions.

Lemma 2.3. Assume that the sequence $0 \to F_1 \to F_2 \to F_3 \to 0$ is exact in $\mathcal{F}$, F_2 is not decomposable and finite. Moreover assume that F_1 and F_3 are simple. Then F_2 lifts to $\mathcal{P}$ if and only if $deg(F_1) = p^k deg(F_3)$ for some integer k.

Proof. Assume first that F_2 lifts to $\mathcal{P}$ and choose $P_2 \in \mathcal{P}$ such that $\iota(P_2) = F_2$. We have two possibilities:

1. The functor P_2 is not simple. Then by exactness of ι and lemma 2 we know that it fits in $\mathcal{P}$ into an exact sequence

$$0 \to P_1 \to P_2 \to P_3 \to 0$$

with simple P_1 and P_3. By the uniqueness of the composition series we know that P_i is a lift of F_i for $i = 1, 3$. Hence $P_1 = F_1^{(i_1)}$ and $P_3 = F_3^{(i_3)}$ by Lemma 1. The functor P_2 has to be indecomposable by exactness of ι and hence $w(P_1) = w(P_3)$. From the description of simple objects in $\mathcal{F}$ we know that

$$p^{i_1} \cdot deg(F_1) = w(P_1) = w(P_3) = p^{i_3} \cdot deg(F_3).$$

This implies the desired relation between $deg(F_1)$ and $deg(F_3)$.

2. The functor P_2 is simple. Let T^j denote the jth tensor product functor. Then by [K3, Section 7]

$$P_2 = T^{j_1} a_1 \otimes T^{j_2 (1)} a_2 \otimes T^{j_3 (2)} a_3 \otimes \ldots \otimes T^{j_s (s-1)} a_s$$

where $a_j \in F_p[\Sigma_j]$ and s is a certain natural number. Let

$$P_2' = T^{j_1(s-1)}a_1 \otimes T^{j_2(s-1)}a_2 \otimes T^{j_3(s-1)}a_3 \otimes ... \otimes T^{j_s(s-1)}a_s$$

Then P_2' lifts F. Now P_2' is not simple and we proceed as in the previous case.

Assume now that $deg(F_1) = p^k \cdot deg(F_3)$ for some integer k. For the argument in this case we will use deep results from [FFSS, Sections 1-3]. Assume that k is non negative, for negative k the proof goes similarly. Let $G_3 = F_3^{(k)}$. Then G_3 is a lift of F_3 and let $d = w(F_1) = w(G_3)$. Let K be an extension of F_p of degree $q \geq d$. For $F \in \mathcal{F}$ *or* $\mathcal{P}$ we write $F(K)$ for the functor in $\mathcal{F}(K)$ or $\mathcal{P}(K)$ obtained from F by scalar extension. We have the following commutative diagram of Ext^1 groups for any field K and any natural number m:

$$\begin{array}{ccccc} K \otimes Ext^1_{\mathcal{P}}(G_3, F_1) & \simeq & Ext^1_{\mathcal{P}(K)}(G_3(K), F_1(K)) & \to & Ext^1_{\mathcal{P}(K)}(G_3(K)^{(m)}, F_1(K)^{(m)}) \\ \downarrow & & \downarrow & & \downarrow \\ K \otimes Ext^1_{\mathcal{F}}(G_3, F_1) & \simeq & Ext^1_{\mathcal{F}(K)}(G_3(K), F_1(K)) & \to & Ext^1_{\mathcal{F}(K)}(G_3(K)^{(m)}, F_1(K)^{(m)}) \end{array}$$

In the upper row the first map is an isomorphism by [FFSS, Proposition 1.1]. The second map is an isomorphism for any m by the solution to the collapsing conjecture given by M.Chalupnik in [C] from which it follows immediately that Frobenius twist induces an isomorphism on $Ext^1_{\mathcal{P}(K)}$ for any K. The vertical maps are induced by ι (the first by $id_K \otimes \iota$).

In the lower row the first map is an isomorphism for large K by [FFSS, Theorem 3.9]. The second map is always an isomorphism by the fact that Frobenius twist is invertible in $\mathcal{F}$. Moreover for large K and m the right vertical map is an isomorphism by [FFSS, Theorem 3.10]. Choosing large enough K and m we get that the first vertical map $id_K \otimes \iota$ is an isomorphism so

$$\iota : Ext^1_{\mathcal{P}}(G_3, F_1) \to Ext^1_{\mathcal{F}}(G_3, F_1)$$

must be an isomorphism also.

Observe that in $\mathcal{F}$, $F_3 = G_3$. Hence an element $\alpha \in Ext^1_{\mathcal{F}}(F_3, F_1)$ corresponding to the extension $0 \to F_1 \to F_2 \to F_3 \to 0$ lifts to $Ext^1_{\mathcal{F}}(G_3, F_1)$. By our previous considerations it lifts to $Ext^1_{\mathcal{P}}(G_3, F_1)$ and hence describes certain functor $P_2 \in \mathcal{P}$ which fits into the exact sequence in $\mathcal{P}$:

$$0 \to F_1 \to P_2 \to G_3 \to 0$$

By the exactness of ι we know that P_2 is a lift of F_2.

We can generalize 2.3 straightforward and try to prove the following statement. Let $F \in \mathcal{F}$ be finite and indecomposable. Let $F_1, ..., F_k$ denote the full list of simple objects obtained as quotients in the composition series for F. Then F lifts to $\mathcal{P}$ iff for any pair (i, j), $deg(F_i) = p^{k_{i,j}} deg(F_j)$ for some integers $k_{i,j}$. Unfortunately the situation is not that simple.

Example 2.4. Let $p = 2$. Then $Ext^1_{\mathcal{F}}(I \otimes I^{(1)}, \Lambda^3) = F_2$ by exponential property of the exterior product. The same formula holds in $\mathcal{P}$ and $Ext^1_{\mathcal{P}}(I \otimes I^{(1)}, \Lambda^3) = F_2 = Ext^1_{\mathcal{F}}(I \otimes I^{(1)}, \Lambda^3)$. Hence there is a functor $F \in \mathcal{F}$ fitting into an exact sequence

$$0 \to \Lambda^3 \to F \to I \otimes I^{(1)} \to 0$$

which does not split. By the equality of Ext^1 groups the functor F lifts to $\mathcal{P}$. On the other hand its composition series consists of functors of degree 2 and 3.

The next example presents a functor in $\mathcal{F}$ which does not lift to $\mathcal{P}$.

Example 2.5. Let $p = 2$ and F_4 denotes the degree 2 field extension of F_2. By [FFSS, Theorem 3.4] we know that

$$F_4 \otimes Ext^1_{\mathcal{F}}(\Lambda^2, \Lambda^3) = Ext^1_{\mathcal{F}(F_4)}(\Lambda^2(I \oplus I^{(1)}), \Lambda^3).$$

The functor $I \otimes I^{(1)}$ is a direct summand in $\Lambda^2(I \oplus I^{(1)})$ so the nontrivial group $Ext^1_{\mathcal{F}(F_4)}(I \otimes I^{(1)}, \Lambda^3)$ embeds in $Ext^1_{\mathcal{F}(F_4)}(\Lambda^2(I \oplus I^{(1)}), \Lambda^3)$. This shows that $Ext^1_{\mathcal{F}}(\Lambda^2, \Lambda^3)$ is nontrivial. Let $F \in \mathcal{F}$ be a functor defined by a nontrivial element in $Ext^1_{\mathcal{F}}(\Lambda^2, \Lambda^3)$. The functors Λ^2 and Λ^3 are simple and all their lifts are given by Frobenius twists. This implies that $Ext^1_{\mathcal{P}}(A, B) = 0$ for A being a lift of Λ^2 and B being a lift of Λ^3 because $w(A) \neq w(B)$. In conclusion, F does not lift to $\mathcal{P}$.

3. A_∞ structures.

Before we start to describe which functors from $\mathcal{F}$ can be lifted to $\mathcal{P}$ we have to compare A_∞-structures related to objects of both categories. We are not going to recall definitions and basic properties of A_∞-structures on DG-algebras and their cohomology algebras, sending readers to [Ka] or [Ke]. Assume that $P \in \mathcal{P}$ and $\iota(P) = F \in \mathcal{F}$. Choose an injective resolution I^* of P in $\mathcal{P}$. Then the cochain algebra $C^* = Hom_{\mathcal{P}}(I^*, I^*)$ is a DG-algebra whose cohomology algebra is equal to $Ext^*_{\mathcal{P}}(P, P)$. We can treat it also as an A_∞-algebra with trivial higher operations (above degree 2). In our case C^* is a DG-algebra with free finite dimensional cohomology over F_p . We can apply the main theorem from [Ka, Theorem 1] which implies directly:

Theorem 3.1: There is an A_∞-structure $(Ext^*_{\mathcal{P}}(P, P), \{X_i\})$ on cohomology of C^* and a morphism of A_∞-algebras $\{f_i\} : (Ext^*_{\mathcal{P}}(P, P), \{X_i\}) \to C^*$ which induces an isomorphism of cohomology algebras.

Let now $J^{*,*}$ be a Cartan-Eilenberg injective resolution of $\iota(I^*)$ in $\mathcal{F}$. $J^{*,*}$ is a bicomplex and we use notation J^* for the cochain complex obtained from $J^{*,*}$ in a standard way. Then $D^* = Hom_{\mathcal{F}}(J^*, J^*)$ is a DG-algebra whose cohomology calculates $Ext^*_{\mathcal{F}}(F, F)$. Similarly to 3.1 we have:

Theorem 3.2: There is an A_∞-structure $(Ext^*_{\mathcal{F}}(F,F),\{Y_i\})$ on cohomology of D^* and a morphism of A_∞-algebras $\{g_i\}:(Ext^*_{\mathcal{F}}(F,F),\{Y_i\})\to D^*$ which induces an isomorphism of cohomology algebras.

As we read in [Ka] the structures of A_∞-algebras on cohomology of DG-algebras are not unique, there are many of them and all of them give isomorphic A_∞-algebras. In our case the functor ι is exact and induces an embedding

$$Ext^*_{\mathcal{P}}(P,P)\hookrightarrow Ext^*_{\mathcal{F}}(F,F)$$

Because of this embedding we will denote cohomology classes in $Ext^*_{\mathcal{P}}(P,P)$ and their images in $Ext^*_{\mathcal{F}}(F,F)$ by the same letters. Our goal in this section is to show the following theorem:

Theorem 3.3: With the notation from theorems 3.1 and 3.2 we can choose operations $\{X_i\}$ and $\{Y_i\}$ in such a way that for any i and any $b_1,b_2,...,b_i\in Ext^*_{\mathcal{P}}(P,P)$ we have

$$X_i(b_1\otimes...\otimes b_i)=Y_i(b_1\otimes...\otimes b_i)$$

.

Proof: This theorem requires some arguments because the Kadeishvili's construction is not unique and consists of series of choices. We have to justify that we can make these choices coherently in $\mathcal{P}$ and $\mathcal{F}$. We will show that performing the inductive construction from [Ka] we can make it in such a way that classes which we choose in D^* are liftings to the resolution of the classes already chosen in C^*. For this we need the following lemma:

Lemma 3.3.1: Let P^* be a cochain complex in an abelian category $\mathcal{C}$ and let $Q^{*,*}$ be a C-E resolution of P^*. Assume that α is a cocycle in $Hom_{\mathcal{C}}(P^*,P^*)$ and $\bar{\alpha}$ is a lift of α to $Hom(Q^{*,*},Q^{*,*})$. Assume that the class of $\bar{\alpha}$ is trivial in cohomology, $\bar{\alpha}=\partial\bar{\beta}$. Then there is $\beta\in Hom_{\mathcal{C}}(P^*,P^*)$ such that $\bar{\beta}$ is a lift of β and $\alpha=\partial\beta$.

Proof of the lemma. Assume that α rises the degree by k, $\alpha_i:P^i\to P^{i+k}$. Assume that for every i, $Q^{i,*}$ is a resolution of P^i. Then P^i embeds into $Q^{i,0}$ and we can directly check that in order to obtain β we have to restrict $\bar{\beta}_{i,0}:Q^{i,0}\to Q^{i+k-1,0}$ to P^i. The result then follows from the fact that $\bar{\alpha}$ lifts α.

Back to the proof of 3.3. We have to recall the Kadeishvili's construction. Let C be a DG-algebra with free cohomology over some field. Kadeishvili constructs the sequence of operations $\{X_i:\otimes^i H(C)\to H(C)\}$ and the sequence of homomorphisms $\{f_i:\otimes^i H(C)\to C$ satisfying certain relations, which show that $\{X_i\}$s give an A_∞-structure on $H(C)$ and $\{f_i\}$s give an A_∞-homomorphism $H(C)\to C$. First he chooses elements of cohomology which present the additive basis of it. Then he defines the structures using only the tensor products of basic vectors and extending definitions by additivity.

The homomorphism f_1 is just a choice of cocycles for cohomology classes and $X_1=0$. When we have defined f_is and X_is for $i<n$ then $X_n(a_1\otimes...\otimes a_n)$ is defined as cohomology

class of a certain directly given cocycle $U_n(a_1 \otimes ... \otimes a_n)$. The formula for $U_n(a_1 \otimes ... \otimes a_n)$ involves f_is and X_is only for $i < n$. The cohomology class of $U_n(a_1 \otimes ... \otimes a_n) - f_1(X_n(a_1 \otimes ... \otimes a_n))$ is trivial and we define $f_n(a_1 \otimes ... \otimes a_n)$ to be the cocycle such that $U_n - f_1(X_n) = \partial f_n$.

We will follow Kadeishvili's construction. We choose first the additive basis $a_1, ..., a_m$ of $Ext^*_{\mathcal{P}}(P,P)$ and we define f_1 as the choice of cocycles in C^* for the basic vectors extended to all cohomology classes by additivity. We define g_1 as the lifting of f_1 to D^*. Of course, as Kadeishvili suggests, we define $X_1 = 0 = Y_1$. Now assume that we have defined X_i, f_i, Y_i and g_i for $i < n$ such that $X_i = Y_i$ and g_i is a lifting of f_i to D^*. Then we see that for any cohomology classes $a_1, ..., a_n$ in $Ext^*_{\mathcal{P}}(P,P)$, $\bar{U}_n(a_1 \otimes ... \otimes a_n)$ is a lifting of $U_n(a_1 \otimes ... \otimes a_n)$ where $\bar{U}_n$ denotes the Kadeishvili's cocycle in D^*. Now we can define f_n and g_n such that g_n is a lifting of f_n using lemma 3.3.1. The described choices give us A_∞-structures as required.

4. Lifting.

Let k be a field and $\mathcal{C}$ be a k-linear abelian category which has enough injective and projective objects. We will always assume in the future that our abelian categories satisfy these conditions without writing this down. Assume that M is an object of $\mathcal{C}$ which has finite filtration with quotients $M_1, ..., M_n$. Our plan is to show that M is described by the data consisting of elements $\alpha_{i,j} \in Ext^1_{\mathcal{C}}(M_i, M_j)$ which are trivial for $j \leq i$ and satisfy certain relation in $Ext^2_{\mathcal{C}}(X,X)$, where $X = M_1 \oplus ... \oplus M_n$ and the relation comes from the A_∞-structure on $Ext^*_{\mathcal{C}}(X,X)$. For this we will use Keller's approach from [Ke] and [Ke1].

One can find in [Ke] and [Ke1] the method of describing a module over an associative algebra via the cohomological data of its quotients. Moreover one can learn there how one can reconstruct various categories of modules from the cohomological information. But Keller's approach is also useful for any small abelian category which can be treated as a full subcategory in the category of modules by the Freyd's theorem. We want to apply Keller's results in the case of categories $\mathcal{F}$ and $\mathcal{P}$ which are F_p-linear and we can treat them as subcategories of the categories of modules over certain F_p-algebras.

Let $D(\mathcal{C})$ denote the derived category of $\mathcal{C}$ and for a finite set of objects $M_1, ..., M_n$ of $\mathcal{C}$ let $tria(M_1, ..., M_n)$ denote the smallest full triangulated subcategory of $D(\mathcal{C})$ which contains all of M_is. Following Keller we denote $filt(M_1, ..., M_n)$ the full subcategory of $\mathcal{C}$ which is the closure under extensions of $M_1, ..., M_n$. We can treat $filt(M_1, ..., M_n)$ as a subcategory of $tria(M_1, ..., M_n)$ in the obvious way. We need the next lemma which is essentially taken from [Ke1] (we send reader to [Ke1] for the definition and properties of derived categories in the context of A_∞-algebras and A_∞-categories).

Lemma 4.1: Let M be an object of an abelian category $\mathcal{C}$ with $M_1, ..., M_k$ the quotients of a certain filtration of M. Let $X = M_1 \oplus ... \oplus M_n$. Then M is defined in the category $tria(M_1, ..., M_n)$ by the sequence of elements $\beta_{ij} \in Ext^1_{\mathcal{C}}(M_i, M_j)$ which are trivial for $j \leq i$ and satisfy certain condition in $Ext^2_{\mathcal{C}}(X,X)$ described by the A_∞-structure on $Ext^*_{\mathcal{C}}(X,X)$

Proof. STEP 1. Let $E = Ext^*_{\mathcal{C}}(X,X)$ be an A_∞-algebra where the A_∞-structure on the Ext-algebra was described (after Kadeishvili) in the previous section. Let $E_i =$

$Ext^*_{\mathcal{C}}(M_i, M_i)$. Section 6 of [Ke1] is devoted to showing that $tria(M_1, ..., M_n)$ is equivalent to $tria(E_1, ..., E_n)$ where this latter category is the smallest triangulated subcategory of the derived category $D_\infty(E)$ of the A_∞-algebra E which contains $E_1, ..., E_n$. This equivalence takes $filt(M_1, ..., M_n)$ to $filt(E_1, ..., E_n)$.

STEP 2. This step is contained in Section 7 of [Ke1]. Following [Ke1, Section 7.7] let $\mathcal{A}$ be an A_∞-category with objects $1, ..., n$ and $\mathcal{A}(i, j) = Ext^*_{\mathcal{C}}(M_i, M_j)$ with composition of morphisms defined via A_∞-operations on $Ext^*_{\mathcal{C}}(X, X)$. Then, by its construction, the category $\mathcal{A}$ is the A_∞-Ext-category of the objects $M_1, ..., M_n$. Let $Y : \mathcal{A} \to C_\infty\mathcal{A}$ be the Yoneda functor. Then, as was shown in step 1, we have an equivalence

$$tria(Y(1), ..., Y(n)) \simeq tria(M_1, ..., M_n)$$

which identifies $filt(Y(1), ..., Y(n))$ with $filt(M_1, ..., M_n)$. The Yoneda functor factorizes through the category $tw\mathcal{A}$ of twisted objects over $\mathcal{A}$, see [Ke1, section 7.5]. Hence every element in $filt(Y(1), ..., Y(n))$ has a description as required by the definition of $tw\mathcal{A}$.

Warning: The reader can complain that our category $\mathcal{A}$ does not satisfy the assumptions of 7.5 and 7.7 because, perhaps, it is not strictly unital. But the results of [Ke1, section 6] do not require this assumption. The factorization of the Yoneda functor into $Y = Y_2 \circ Y_1$, as described in [Ke1, 7.5], also works in full generality. Strict unitality is necessary for the conclusion that $H^0(tw\mathcal{A})$ is equivalent to $filt(Y(1), ..., Y(n))$. But we do not need such a strong statement. Of course we have to pay a price for this. Our final result tells only that a functor $F \in \mathcal{F}$ satisfying certain conditions can be lifted from $\mathcal{F}$ to $\mathcal{P}$. For a given $F \in \mathcal{F}$, which is not simple, the number of lifts is still not known and cannot be calculated by our methods.

Lemma 4.2: Let $M_1, ..., M_k$ be objects of an abelian category $\mathcal{C}$ and let $\beta_{i,j} \in Ext^1_{\mathcal{C}}(M_i, M_j)$ for $1 \le i, j \le k$. Assume that:
- $\beta_{i,j} = 0$ for $j \le i$.
- the elements $\beta_{i,j}$, $1 \le i, j \le k$, satisfy the condition defining a twisted object ([Ke, Section 7.6, formulae 7.1]).

Then there exists $M \in \mathcal{C}$ for which this is the defining data in $tria(M_1, ..., M_k)$ in the sense of lemma 4.1.

Proof. Let $\mathcal{A}$ be as in Step 2 above and let Z be a twisted object in $tw\mathcal{A}$ defined by the elements $\beta_{i,j}$, $1 \le i, j \le k$ and non shifted copies of objects $1, ..., n$. Following [Ke1, Section 7.5] let $C_\infty\mathcal{A}$ be the category of $\mathcal{A}$-modules and $Y_2 : tw\mathcal{A} \to C_\infty\mathcal{A}$ be a factorization of the Yoneda functor through $tw\mathcal{A}$. Then we can easily check that the object $Y_2(Z)$ belongs to $filt((Y(1), ..., Y(n))$ and hence defines an object M in $filt(M_1, ..., M_n)$. The defining data for M in the sense of 4.1 is as required.

Now we come back to functor categories. Let F be an indecomposable finite object in $\mathcal{F}$. We can restrict our attention to indecomposable objects by obvious reasons. Assume that F has in $\mathcal{F}$ a filtration with quotients $F_1, ..., F_k$ satisfying for any i, $F_i = \iota(P_i)$ for a certain simple object in $\mathcal{P}$. Then, accordingly to 4.1, F is described by the set of elements

$\beta_{i,j} \in Ext^1_{\mathcal{F}}(F_i, F_j)$, which are trivial for $j \leq i$ and satisfy certain relation in $Ext^2_{\mathcal{F}}(.,.)$. We have the following theorem:

Theorem 4.3: The functor F lifts to $\mathcal{P}$ iff all elements $\beta_{i,j}$ lift to $\alpha_{i,j} \in Ext^1_{\mathcal{P}}(P_i, P_j)$.

Proof. The functor ι is exact, it induces a monomorphism on Ext-groups. The results of Section 3 tell us that it induces well defined functor between the categories of twisted objects in $\mathcal{P}$ and $\mathcal{F}$. If $F = \iota(P)$ for some $P \in \mathcal{P}$ then it has the desired filtration with quotients coming from the composition series of P. The existence of elements $\beta_{i,j}$ follows from Lemma 4.1. They lift to themselves after identification of classes from $Ext^1_{\mathcal{P}}(P_i, P_j)$ with classes in $Ext^1_{\mathcal{F}}(F_i, F_j)$ as in Section 3.

But we are really interested in the opposite implication. Let $\mathcal{A}_{\mathcal{P}}$ ($\mathcal{A}_{\mathcal{F}}$) be the A_∞-category as in in Step 2 of the proof of 4.1 for $P_1, ..., P_k$ ($F_1, ..., F_k$). Assume that for any i, j , $\beta_{i,j}$ lifts to $\alpha_{i,j} \in Ext^1_{\mathcal{P}_d}(P_i, P_j)$. The functor ι defines an A_∞-functor (identity on objects) denoted by the same letter

$$\iota : \mathcal{A}_{\mathcal{P}} \to \mathcal{A}_{\mathcal{F}}$$

because by the results of Section 3 the A_∞-structures can be chosen to be compatible. It extends to a functor

$$\iota : tw\mathcal{A}_{\mathcal{P}} \to tw\mathcal{A}_{\mathcal{F}}$$

because the map induced by ι on Ext^2 is a monomorphism. The data $\{\alpha_{i,j}\}$ as above defines an object $Z_{\mathcal{P}}$ in $tw\mathcal{A}_{\mathcal{P}}$ and ι takes it to the twisted object $Z_{\mathcal{F}}$ in $tw\mathcal{A}_{\mathcal{F}}$ defined by $\{\beta_{i,j}\}$. Let P be an object in $\mathcal{P}$ corresponding to $Y_2(Z_{\mathcal{P}})$. By construction, the defining data for for $\iota(P)$ is given by the classes $\beta_{i,j}$. Hence P is a lifting of F.

5. Final remarks.

In this section we want to present two problems concerning cohomological behavior of finite functors in $\mathcal{F}$ which were studied before. We can easily prove them for functors in the image of ι and they are open in general. Throughout the section we assume that our functors $F \in Ob(\mathcal{F})$ are *finite* what means that they are of finite degree in the sense of Eilenberg and Mac Lane and take finite dimensional values.

Problem 5.1: Prove that for any finite $F \in Ob(\mathcal{F})$ the vector space $Ext^*_{\mathcal{F}}(I, F)$ is either trivial or infinite dimensional.

Theorem 5.2: If F can be lifted to $\mathcal{P}$ then 5.1 holds for F.

Proof. We can obviously assume that F is indecomposable by the additivity of the Ext-groups. This implies that there is an integer t and a strict polynomial functor $P \in Ob(\mathcal{P}_t)$ of homogeneous degree t such that $\iota(P) = F$. The functor P has in $\mathcal{P}_t$ a finite injective resolution $P \to Q_0 \to Q_1 \to ...$ in which every Q_j is a sum of tensor products of symmetric powers:

$$Q_j = \bigoplus S^{j_1} \otimes ... \otimes S^{j_m}$$

where $j_1 + ... + j_m = t$. This resolution remains exact in $\mathcal{F}$ after applying ι. If $m > 1$ then $S^{j_1} \otimes ... \otimes S^{j_m}$ is obviously diagonalizable. If $m = 1$ and j_1 is not a power of p then S^{j_1} is a direct summand in a diagonalizable functor (see [FLS, Proposition 6.1]). Hence by a hyper-cohomology spectral sequence argument and vanishing of $Ext^*_{\mathcal{F}}(I, .)$ for diagonalizable functors we know that $Ext^*_{\mathcal{F}}(I, F)$ can be non trivial only when t is a power of p. We will assume that $t = p^i$ for the rest of the proof.

Let $F^{\#}$ denotes the Kuhn's dual of F and $V^{\bullet}$ be the liner dual of the vector space V. The crucial argument for the proof of 5.2 is taken from [C1] where Chalupnik proved the following theorem ([C, Theorem 3.2]):

Theorem: Let $P \in \mathcal{P}_d$ be simple and projective, or let $P = I^d$ (the dth tensor power functor). Then for any $s \geq 0$ we have a natural in F isomorphism

$$Ext^s_{\mathcal{P}(k)_{p^i d}}(P^{(i)}, F) \simeq Ext^{2(p^i-1)d-s}_{\mathcal{P}(k)_{p^i d}}(P^{(i)}, F^{\#})^{\bullet}.$$

Observe first that the Frobenius twist is equal to identity on $\mathcal{F}$ and hence for any natural numbers j and m we have:

$$Ext^j_{\mathcal{F}}(I, F) = Ext^j_{\mathcal{F}}(I^{(i)}, F) = Ext^j_{\mathcal{F}}(I^{(i+m)}, F^{(m)})$$

Remembering that P is of homogeneous degree t assume that $\bar{\alpha} \in Ext^j_{\mathcal{F}}(I^{(i)}, F)$ is nontrivial and comes from $\alpha \in Ext^j_{\mathcal{P}}(I^{(i)}, P)$ by applying ι. By Chalupnik's theorem we detect class $\beta \in Ext^{2(p^i-1)-j}_{\mathcal{P}}(I^{(i)}, P^{\#})$ corresponding to α. The choice of β is obviously not unique. Frobenius twist induces a monomorphism on $Ext_{\mathcal{P}}$ by [FFSS, corollary 1.3]. It means that for any natural m, β defines nontrivial classes

$$\beta_m \in Ext^{2(p^i-1)-j}_{\mathcal{P}}(I^{(i+m)}, P^{(m)\#}).$$

Using again the Chalupnik's result and the fact that $(P^{\#})^{\#} = P$ we get classes $\alpha_m \in Ext^{2p^i(p^m-1)+j}_{\mathcal{P}}(I^{(i+m)}, P^{(m)})$. Because i and j are fixed and m is arbitrary, the number $2p^i(p^m-1)+j$ can be arbitrarily large. Hence we know that classes α_m define infinite sequence of nontrivial cohomology classes which are contained in different degrees of *Ext*-groups. By [FFSS, Corollary 3.7] we know that the map induced by ι:

$$Ext^{2p^i(p^m-1)+j}_{\mathcal{P}}(I^{(i+m)}, P^{(m)}) \longrightarrow Ext^{2p^i(p^m-1)+j}_{\mathcal{F}}(I^{(i+m)}, F^{(m)}) = Ext^{2p^i(p^m-1)+j}_{\mathcal{F}}(I, F)$$

is injective. This implies our theorem under the assumption that every class $\bar{\alpha}$ come from α as above.

For our purposes it is enough to show that for any nontrivial $\bar{\alpha} \in Ext^j_{\mathcal{F}}(I^{(i)}, F)$ there exists m such that $\bar{\alpha}$ comes from $\alpha \in Ext^j_{\mathcal{P}}(I^{(i+m)}, P^{(m)})$. In order to show this we have to use more results from [FFSS]. Let K be a finite extension of F_p. We will write $\mathcal{F}(K)$ ($\mathcal{P}(K)$) and F_K (P_K) for the category o functors over the base field K and for the functor obtained from F (P) by the scalar extension. We list the needed results:

1.

$$K \otimes Ext^j_{\mathcal{P}}(I^{(i)}, P) = Ext^j_{\mathcal{P}(K)}(I^{(i)}, P_K)$$

Proof: [FFSS, Proposition 1.1].

2.

$$K \otimes Ext^j_{\mathcal{F}}(I^{(i)}, F) = Ext^j_{\mathcal{F}(K)}(I^{(i)}, F_K)$$

Proof: By [FFSS, Remark 3.4.1] we know that

$$K \otimes Ext^j_{\mathcal{F}}(I^{(i)}, F) = Ext^j_{\mathcal{F}(K)}(I^{(i)} \circ (t \circ \tau), F_K).$$

Here t is the scalar extension functor and τ is the forgetful functor, as studied in [FFSS, Section 3]. Now using the scalar decomposition of the category $\mathcal{F}(K)$ described in [K, Section 3.3] we get immediately that this latter group is equal to $Ext^j_{\mathcal{F}(K)}(I^{(i)}, F_K)$.

3. For given j there is a field extension K of F_p and a number m such that

$$K \otimes Ext^j_{\mathcal{P}(K)}(I^{(i+m)}, P^{(m)}) = Ext^j_{\mathcal{F}(K)}(I^{(i)}, F_K)$$

Proof: [FFSS, Theorem 3.10].

Now we can consider the following commuting (see [FFSS, Theorem 3.5]) diagram:

$$\begin{array}{ccc} K \otimes Ext^j_{\mathcal{F}}(I, F) & \simeq & Ext^j_{\mathcal{F}(K)}(I^{(i)}, F_K) \\ \uparrow & & \uparrow \\ K \otimes Ext^j_{\mathcal{P}}(I^{(m+i)}, F^{(m)}) & \simeq & Ext^j_{\mathcal{P}(K)}(I^{(i+m)}, F^{(m)}_K) \end{array}$$

where the right-hand vertical arrow is an isomorphism. This implies our claim and the proof of the theorem is finished.

Now we come to the second problem. Let QF denote the Mac Lane's Q-construction related to F, defined and developed in [J-M, Sections 6 and 7]. By definition QF is a nonnegative chain complex in $\mathcal{F}$ whose homology functors are isomorphic to the left stable derived functors of F. It has $(QF)_0 = F$ and hence it comes with the map of chain complexes $F \to QF$ where, as usual, F is treated as a chain complex concentrated in dimension 0. This map induces for any i a homomorphism of hyperext groups

5.3.1

$$\mathbf{Ext}^i_{\mathcal{F}}(QF, I) \to \mathbf{Ext}^i_{\mathcal{F}}(F, I)$$

5.3.2

$$\mathbf{Ext}^i_{\mathcal{F}}(I, F) \to \mathbf{Ext}^i_{\mathcal{F}}(I, QF)$$

The map from 5.3.1 is easily seen to be an isomorphism by the spectral sequence argument for the hyperext groups. In [S] Stefan Schwede showed that 5.3.2 was an isomorphism for symmetric powers and used that result for interesting applications in homotopy theory. It is given in [S, 11.3] an example of a functor in $\mathcal{F}$ of infinite degree for which $\mathbf{Ext}^0(I, F)$ and $\mathbf{Ext}^0(I, QF)$ really differ.

Problem 5.3: Show that the homomorphism 5.3.2 is an isomorphism for any finite $F \in Ob(\mathcal{F})$.

Theorem 5.4: Problem 5.3 has positive solution for any F which can be lifted to $\mathcal{P}$.

Proof. The Q construction takes short exact sequences of functors to short exact sequences of chain complexes. From this we see that if

$$0 \to F_1 \to F_2 \to F_3 \to 0$$

is a short exact sequence in $\mathcal{F}$ than if the conclusion of the theorem holds for two of $\{F_1, F_2, F_3\}$ then it holds also for the third by the long exact sequence argument. Assume that $F \in Ob(\mathcal{F})$ is finite and can be lifted to $\mathcal{P}$, $F = \iota(P)$ for a certain $P \in Ob(\mathcal{P})$. We can argue like in 5.2 and reduce ourselves to the case when F is indecomposable of homogeneous degree $t = p^i$. As previously, if this is the case then there exists in $\mathcal{F}$ a finite exact sequence $F \to Q_0 \to Q_1 \to ... \to Q_k$ in which every Q_j is a sum of tensor products of symmetric powers:

$$Q_j = \bigoplus S^{j_1} \otimes ... \otimes S^{j_m}$$

where $j_1 + ... + j_m = t$.

Observe that for all Q_js our theorem holds either by the results of Schwede or by vanishing of both sides in 5.3.2. We will prove our theorem easily by induction on k. Very briefly: let G denote the cokernel in $\mathcal{F}$ of the map $F \to Q_0$. Then G lifts to $\mathcal{P}$, more precisely $G = \iota(P_G)$ where $P_G = coker(P \to Q_0)$. By construction P_G has shorter injective resolution in $\mathcal{P}$ than P. Hence by inductive hypothesis our theorem holds for G. It holds for Q_0 so it holds also for F.

Bibliography:

[C] - M. Chalupnik. *Derived Kan extension for strict polynomial functors.* Int. Math. Res. Not. 20 (2015) 1001710040.

[C1] - M. Chalupnik. *Poincaré duality for Ext-groups between strict polynomial functors.* Proc. Amer. Math. Soc. 144 (2016) 963-970.

[FFSS] - V. Franjou, E. Friedlander, A. Scorichenko, A. Suslin. *General linear and functor cohomology over finite fields.* Annals of Math. 150 (1999) 663-728.

[JM] - B. Johnson, R. MacCarthy: *Linearization, Dold-Puppe stabilization and Mac Lane's Q-construction.* TAMS 350 (1998) 1555-1593.

[FLS] - V. Franjou, J. Lannes, L. Schwartz. *Autour de la cohomologie de MacLane des corps finis.* Invent. Math. 115 (1994) 513-538.

[Ka] - T. V. Kadeishvili. *On the theory of homology of fiber spaces.* Uspekhi Mat. Nauk 35 (1980) 183-188. Translated in Russ. Mat. Surv. 35 (1980) 231-238.

[Ke] - B. Keller. *$\mathcal{A}$-infinity algebras in representation theory.* Representations of algebra. Vol. I, II, 7486, Beijing Norm. Univ. Press, Beijing, 2002.

[Ke1] - B. Keller. *Introduction to A-infinity algebras and modules.* Homology, Homotopy and Appli. 3 (2001) 1-35.

[K] - N. Kuhn. *Generic representations of the finite general linear groups and the Steenrod algebra: I.* American Journal of Mathematics 116 (1994) 327-360.

[K1] - N. Kuhn. *Generic representations of the finite general linear groups and the Steenrod algebra: II.* K-Theory 8 (1994), 395428.

[K2] - N. Kuhn. *Generic representations of the finite general linear groups and the Steenrod algebra: III.* K-theory 9 (1995) 273-303.

[K3] - N. Kuhn. *A stratification of generic representation theory and generalized Schur algebras.* K-theory 26 (2002) 15-49.

[S] - S. Schwede. *Formal groups and stable homotopy of commutative rings.* Geom. Topol. 8 (2004), 335-412.

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