Riemann zeta via λ-rings

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ABSTRACT: We define the field F_1 of one element as a λ-ring $\mathbf{Z}$ with the canonical λ-structure. We show that we can calculate the Riemann zeta function of integers in two ways: the first, geometrical, as the zeta function of the affine line $F_1[x]$ over F_1 and the second, categorical, using a suitable category of modules over F_1.

0. Introduction.

We would like to adopt the concept of Borger from [B] to study the Riemann ζ-function of integers. Borger claims, that the category of rings over F_1 should consist of λ-rings and the restriction of scalars from $\mathbf{Z}$ to F_1 takes any commutative ring R to its ring of Witt vectors $W(R)$ with the canonical λ-structure. In this approach the mythical field F_1 is equal to the ring of integers $\mathbf{Z}$ with the canonical λ-structure. We will denote it as $\mathbf{Z}_\lambda$ or simply F_1, when it will be clear which model of F_1 we have in mind.

We proved in [Be] that the categorical ζ-function of the category of commutative monoids calculates the Riemann ζ-function of the integers. This was done in order to support the idea, that while trying to view $\mathbf{Z}$ as a variety over the field with one element we should consider integers as multiplicative monoid without addition. The idea that $\mathbf{Z}$ treated as a variety over F_1 should live in the category of monoids is well described in the literature, see for example [KOW] or [D]. But, because category of monoids is too rigid, most authors instead of working with monoids directly extend their field of scalars from F_1 to $\mathbf{Z}$ (or other rings), assuming that scalar extension from F_1 to $\mathbf{Z}$ should take a monoid A to its monoid ring $\mathbf{Z}[A]$. This agrees well with the expectation that rings should be treated in the category of monoids as monoids with ring multiplication as a monoidal operation. Then the forgetful functor from rings to monoids and the scalar extension as described above give us the nice pair of adjoint functors. But this approach carries one disadvantage. It takes us quickly from formally new approach via monoids to the classical world of rings and modules over them or to other abelian categories. Working with λ-rings seems to be a good way of overcoming this disadvantage. We will preserve monoidal point of view but our approach will allow to use more algebra-geometric methods then the ones which are at hand in the world of monoids.

The definition of F_1 as $\mathbf{Z}$ with the canonical λ-structure does describe this object almost like a field. If λ-operations are part of the structure then the ideals in our rings should be preserved by them. It is easy to observe that in $\mathbf{Z}_\lambda$ there are no proper ideals preserved by λ-operations. Our plan is to justify the following observation that in the category of λ-rings we can calculate the Riemann ζ_R of the integers in two ways. One using the suitable category of modules over $\mathbf{Z}_\lambda$ and the other (geometric) via calculating numbers of fixed points of the action of the Frobenius morphism on the affine line over the algebraic closure of $\mathbf{Z}_\lambda$.

In order to show that our program works we will try first of all to answer the question what should be the algebraic closure of F_1 and the Frobenius action on $\bar{F}_1$. Then we show that the ζ-function of the affine line over F_1 calculated via counting the fixed points of the Frobenius action is the same as the ζ-function of integers. Next we have to find the proper category of modules over F_1 or, in general, over any λ-ring and in this respect we follow the lines of [H, chapter 2]. At the end we show how to calculate the Riemann ζ-function using the category of modules over F_1.

I. F_1 and the category of commutative monoids.

Let M_{ab} denote the category of commutative monoids with unit and unital maps. This section is devoted only to the rough description of certain features of M_{ab} which are crucial for our approach in the next sections. They are either obvious or well described elsewhere so we are very brief here.

Let us start from a some piece of notation. We will denote by $\mathbf{N}^*$ the multiplicative monoid of natural numbers. The symbol $\mathbf{N}^+$ denotes the monoid of natural numbers with addition. Observe that any monoid $M \in M_{ab}$ carries natural action of $\mathbf{N}^*$ by identifying $k \in \mathbf{N}^*$ with $\psi^k : M \to M$ where $\psi^k(m) = m^k$. This structure is obvious in M_{ab} and adds very little while studying monoids, but should be reflected always, when we want to induce structures from M_{ab} to other (abelian) categories. This action will be addressed as an action of Adams operations on M. We can also interpret it as an action of the powers of the Frobenius endomorphism. Observe that in characteristic p the Frobenius endomorphism acts by rising an element to the pth power. We propose to view the Frobenius action in a uniform, characteristic free way. In any structure by Frobenius action we mean an action of $\mathbf{N}^*$, where $k \in \mathbf{N}^*$ acts by rising an element to the kth power if this gives a morphism in the considered structure.

In [D] Deitmar developed the algebraic geometry in the setting of monoids, associating to $M \in Obj(M_{ab})$ a topological space with a sheaf of monoids which resembles a spectrum of prime ideals of a commutative ring with its structural sheaf. In his language F_1 is the one element monoid consisting only of the unit.

Definition 1.1: For any unital monoid M we define the polynomial ring over M by

$$M[x] = M \times \mathbf{N}^+.$$

We write ax^n for the element (a, n) with convention $(a, 0) = a$. The evaluation of ax^n at the point $b \in M$ equals ab^n.

Remark 1.2: Our definition agrees with the general expectation (compare [D], [S]) that scalar extension from F_1 to $\mathbf{Z}$ should take any monoid A to its integral monoid ring $\mathbf{Z}[A]$. Indeed we have an isomorphism of rings

$$\mathbf{Z}[M \times \mathbf{N}^+] \to \mathbf{Z}[M][x]$$

given by the formula

$$\sum_{i\in I} z_i(a_i, n_i) \mapsto \Sigma z_i a_i x^{n_i}$$

Since we know what is the polynomial ring over a monoid we can try to imagine what is the algebraic closure of F_1.Typically, when we want to close algebraically a field K we have to add to it, at least, all roots of elements of K (this is sufficient in the finite field case). In other words we can say that $\bar{K}$ is the minimal field which contains all roots of the elements of K and has no algebraic extensions. This second condition is equivalent to saying that all roots of elements of $\bar{K}$ are contained in $\bar{K}$. So it should not be of great surprise to propose:

Claim 1.3: $\bar{F}_1 = \mathbf{Q}/\mathbf{Z}$

For any monoid M we write M_+ for M with 0 added to it. This is important when we want to talk about geometric points of varieties over F_1. In [D] prime ideals in M are given by subsets of M satisfying obvious for ideals conditions. Empty set is a good prime ideal, and it is necessary to consider it at least in order to have one point in $Spec(F_1)$. When we have a variety M over F_1 (a monoid) and B is an extension of F_1 we can talk about $M(B)$, the B-points of M. They are equal (following [D]) to $Hom_{M_{ab}}(M_+, B_+)$. Here we have to add zeros to monoids because a map cannot have an empty value on some element.

We can calculate the number of points of $\mathbf{N}^+$, treated as an affine line over F_1, in $\bar{F}_1$. The Frobenius action on $\mathbf{Q}/\mathbf{Z}$ extends to the action on $\mathbf{Q}/\mathbf{Z}$-points of $\mathbf{N}^+$. We easily calculate that for $k \in \mathbf{N}^*$, $k \neq 1$ we have the following formula for the order of the set of fixed points:

$$|(\mathbf{N}^+(\mathbf{Q}/\mathbf{Z}))^k| = k$$

If we view the Weil zeta function for varieties over F_p as a way of keeping track of the numbers of their points over $\bar{F}_p$ fixed by the action of the powers of the Frobenius morphism then we get the formula for the Riemann zeta function ζ_R in the similar spirit in M_{ab}:

$$\zeta(\mathbf{N}^+, s) = 1 \ + \ \sum_{k=2}^{\infty} |(\mathbf{N}^+(\mathbf{Q}/\mathbf{Z}))^k|^{-s}$$

As we said in the introduction the geometry in M_{ab} is too weak to expect that we can approach the Riemann conjecture using it. But the picture described above is very basic and leads to the correct ζ function. This implies that we want to see our more sophisticated structures of F_1-objects as the structures induced from the picture described above.

II. Preliminaries on λ-rings.

Our rings are always commutative with units. Following [Y, Definition 1.10] we have:

Definition 2.1: A λ-ring is a ring R together with functions

$$\lambda^n : R \to R \quad (n \geq 0)$$

satisfying for any $x, y \in R$:

(1) $\lambda^0(x) = 1$,
(2) $\lambda^1(x) = x$,
(3) $\lambda^n(1) = 0$ for $n \geq 2$,
(4)$\lambda^n(x+y) = \sum_{i+j=n} \lambda^i(x)\lambda^j(y)$,
(5) $\lambda^n(xy) = P_n(\lambda^1(x), ..., \lambda^n(x); \lambda^1(y), ..., \lambda^n(y))$,
(6) $\lambda^n(\lambda^m(x)) = P_{n,m}(\lambda^1(x), ..., \lambda^{nm}(x))$.

Above P_n and $P_{n,m}$ are certain universal polynomials with integer coefficients obtained via symmetric functions theory (see [Y, Example 1.7 and 1.9]). By a homomorphism of λ-rings we mean a ring homomorphism which commutes with λ-operations. We say that $x \in R$ is of degree k, if k is the largest integer for which $\lambda^k(x) \neq 0$. If such finite k does not exist we say that x is of infinite degree. Observe that (by formula 4) the map

$$R \ni x \mapsto \sum_{i \geq 0} \lambda^i(x)t^i$$

is a homomorphism from the additive group of R to the multiplicative group of power series over R with constant term 1. We will denote this map as $\lambda_t(x)$. Observe also that $\lambda_t(0) = 1$ and hence $\lambda_t(-r) = \lambda_t(r)^{-1}$. Usually in the literature the set of power series over R with constant term 1, with addition defined by the power series multiplication and with properly defined multiplication is called the universal λ-ring of R and denoted $\Lambda(R)$ (see [Y, Chapter 2] for the full discussion on this concept). The universal Λ-ring of R can be defined for any commutative ring R but when R is a λ-ring then $\lambda_t : R \to \Lambda(R)$ is a λ-ring homomorphism.

Ring of integers $\mathbf{Z}$ carries the unique, canonical λ-ring structure described by the formula $\lambda^n(m) = \binom{m}{n}$. Similarly all integral monoid rings $\mathbf{Z}[M]$ will be considered with the λ structure defined for any $m \in M$ by formulas

$$\lambda^1(m) = m$$

$$\lambda^i(m) = 0 \ \text{ for } \ i > 1.$$

We will always consider integral monoid rings with such λ-structure, because this structure is uniquely forced by the monoidal point of view on the field with one element (compare Proposition 2.3 and the discussion which follows it). In most of our monoids the cancelation law will hold. It means that typically if $x, y, z \in M$ and $xy = xz$ then $y = z$. We have:

Lemma 2.2: Let R be equal to the monoidal λ-ring $\mathbf{Z}[M]$ with the λ-structure defined above. Assume that cancelation law holds in M. Then in R only generators $m \in M \subset \mathbf{Z}[M]$ are of degree 1.

Proof. We know that the elements of M are of degree 1. By separating positive and negative coefficients we get $\mathbf{Z}[M] \ni r = \Sigma_{i=1}^{k} a_i m_i - \Sigma_{j=1}^{l} b_j m_j$, where all a_is and b_js belong to $\mathbf{Z}$ and are greater than 0. Observe that the assumption that r is of degree 1 implies that $\lambda_t(r) = 1 + bt$ for a certain $b \in \mathbf{Z}[M]$. We can easily calculate λ_t-functions in the case of monoidal rings. Let $m \in M$ and a be a positive integer. Then

$$\lambda_t(am) = \lambda_t(m)^a = (1 + mt)^a$$

Hence we easily get

$$\lambda_t(\sum_{i=1}^{k} a_i m_i - \sum_{j=1}^{l} b_j m_j) = \prod_{i=1}^{k}(1 + m_i t)^{a_i} / \prod_{j=1}^{l}(1 + m_j t)^{b_j}$$

If r is of degree 1 we have equality

$$(*) \quad \prod_{i=1}^{k}(1 + m_i t)^{a_i} = (1 + bt) \prod_{j=1}^{l}(1 + m_j t)^{b_j}$$

From this, by comparing coefficients at the highest degree of t we get

$$b = \prod_{i=1}^{k}(m_i)^{a_i} / \prod_{j=1}^{l}(m_j)^{b_j}$$

and hence $b \in M$. On the other hand, when we calculate the coefficient at the first degree in the equality **(*)** we get

$$\sum_{i=1}^{k} a_i m_i = b + \sum_{j=1}^{l} b_j m_j$$

But by the definition of a_is and b_js this is possible only when $r = b \in M$.

Recall that M_{ab} denotes the category of commutative monoids with unit. Let $Ring^{\lambda}$ stand for the category of commutative unital rings with λ-structure. We have:

Proposition 2.3: The functor $M_{ab} \to Ring^{\lambda}$, which takes a monoid M to the λ-ring $\mathbf{Z}[M]$ with the λ-structure defined above has a right adjoint $Ring^{\lambda} \to M_{ab}$ which takes a λ-ring R to the multiplicative monoid R_1 of its elements of degree not exceeding 1.

Proof. By [Y, Proposition 1.13] we know that in any λ-ring the product of 1-dimensional elements is again 1-dimensional (or equal to 0). Hence R_1 is a well defined multiplicative submonoid of R considered here as the multiplicative monoid. If $f \in Mor_{Ring^{\lambda}}(R, S)$ then f carries 1 dimensional elements of R to 1 dimensional elements of S or to 0 by the definition of a λ-homomorphism. Hence our right adjoint is well defined. The rest of the proof is obvious.

The λ-operations on a ring R define on it the sequence of Adams operations $\psi^k : R \to R$ which are natural ring homomorphisms. They can be defined by the Newton formula:

$$\psi^k(x) - \lambda^1(x)\psi^{k-1}(x) + ... + (-1)^{k-1}\lambda^{k-1}(x)\psi^1(x) = (-1)^{k-1}k\lambda^k(x)$$

For their properties see [Y, chapter 3]. It is straightforward to check that the canonical λ-structure on $\mathbf{Z}$ defines trivial Adams operations ($\psi^k = id$ for any k) and the formula $m \mapsto m^k$ for $m \in M$ determines the kth Adams operation on the monoidal ring $\mathbf{Z}[M]$. Adams operations can be viewed always as an action on a considered structure by the multiplicative monoid $\mathbf{N}^*$ of natural numbers. Every object M of M_{ab} has naturally such a structure as was described in Section 1. So proposition 2.3 can be viewed as a statement about adjoint functors between categories with objects carrying the action of $\mathbf{N}^*$. It is easy to check that the $\mathbf{N}^*$ action on $\mathbf{Z}[M]$ given by $k(m) = m^k$ while treated as the action of Adams operations forces to have λ-structure on $\mathbf{Z}[M]$ satisfying $\lambda^i(m) = 0$ for $i > 1$.

Recall that $\mathbf{N}^+$ denotes the monoid of natural numbers with addition. Using it we can define for any ring R the polynomial ring over R via the formula

$$R[x] = R[\mathbf{N}^+] = R \otimes \mathbf{Z}[\mathbf{N}^+].$$

We will consider $Z[\mathbf{N}^+]$ as polynomial ring over F_1 in the rest of the paper, with λ-structure defined like for any other monoidal ring. Moreover, for any λ-ring R we have well defined λ-structure on $R[x]$ because tensor product of rings inherits it from the λ-structures of the factors.

REMARK: As we said before the λ-structure on $\mathbf{Z}$ is unique. This is not the case with monoidal rings $\mathbf{Z}[M]$. We want to view the results of Section I and our Proposition 2.3 as strong indication, that the restriction of considered λ structures on our monoidal rings to the ones defined by Adams operations on monoids is well justified by the results of [Be]. This restriction is crucial for the whole of the paper and more generally for the whole of our approach to studying the Riemann ζ-function of integers.

Definition 2.4: Let R be a λ-ring and I is an ideal in R. We will call it a λ-ideal if it is preserved under the action of λ^k, for any $k > 0$.

It is straightforward to check that if we divide a λ-ring by a λ-ideal then R/I carries the induced λ-structure and the quotient homomorphism $R \to R/I$ is a homomorphism of λ-rings. Of course the opposite is also true: a kernel of the λ-rings homomorphism is a λ-ideal. It is important for computations that an ideal I in a λ-ring R with $\mathbf{Z}$-torsion free quotient is a λ-ideal if and only if it is preserved by the Adams operations (see [Y, Corollary 3.16]).

III. Algebraic extensions of F_1 and its algebraic closure $\bar{F}_1$.

As we said in the introduction, the λ-ring $\mathbf{Z}_\lambda$ (our hypothetical F_1) can be treated as a field because it contains no proper λ-ideals. We have defined the ring of polynomials over F_1. On the other hand we can always view an algebraic closure of a field via the ring of polynomials and its quotients. If K is a field we know that every algebraic extension of K should be contained in $\bar{K}$. We know that every algebraic extension of K is build out of simple extensions $K \subset K(a)$ where a is a root of a non-decomposable polynomial $f_a \in K[x]$. Moreover, for simple extensions we have the formula $K(a) = K[x]/(f_a)$. It means that we can view $\bar{K}$ as a sum of the fields of the form $L[x]/(f)$ where L is an algebraic extension of K or even as a sum of simple extensions $K[x]/(f)$. We will try to use this point of view in our context but remembering all the time about the λ-structures on our objects. Let us start from the following lemma:

Lemma 3.1: Every monic polynomial f, which generates the principal λ-ideal in $F_1[x]$ has only roots of unity or 0 as his roots in $\mathbf{C}$. If $f(\mu_n) = 0$, where μ_n is the prime root of 1 of degree n, then $x^n - 1 \mid f$ (and hence $n \leq deg(f)$).

Proof. Because our polynomials are monic the quotients of $F_1[x]$ by ideals generated by them are torsion free. Hence instead of working with λ-operations we can assume that our ideals are preserved by all Adams operations. Assume that an ideal I is generated by a polynomial $f(x) = x^n + a_{n-1}x^{n-1} + ... + a_0$. Then $\psi^k(x^n + a_{n-1}x^{n-1} + ... + a_0) = x^{kn} + a_{n-1}x^{k(n-1)} + ... + a_0$. This formula implies that if a is a root of f then a^k is also a root of f for any natural k, when we calculate roots in the field of complex numbers. To see this observe that the evaluation at a map $e_a : F_1[x] \to \mathbf{C}$ is a homomorphism of rings. Let $I = ker(e_a)$ then $(f) \subset I$ and because $\psi^k((f)) \subset (f)$ then $\psi^k(f) \in (f) \subset I$. But for a polynomial f the statement that if a is a root of f then a^k is also a root for any natural k implies directly that a is a root of 1 or $a = 0$. Hence all the zeros of f are the roots of unity or are equal to 0. On the other hand if a is a primitive root of 1 of degree s and $f(a) = 0$ then all the other roots of unity of degree s are among the zeros of f, again by the argument with Adams operations. This implies immediately our statement.

Definition 3.2: Let K be a λ-ring. We will say that $f \in K[x]$ is non-decomposable if the principal ideal (f) is preserved by the λ-operations and there is no decomposition $f = f_1 \cdot f_2$ such that $deg(f_1) > 0$, $deg(f_2) > 0$ and both ideals (f_1) and (f_2) are preserved by the λ-operations.

Lemma 3.1 describes what are the non-decomposable polynomials in $F_1[x]$. In our approach we would like to take the following definition for an algebraic extension in the category of λ-rings (field extensions of F_1).

Definition 3.3: Let F be a λ-ring without nilpotent elements. We say that:

i. $K \supset F$ is a simple algebraic extension of F if K is isomorphic to $F[x]/(f)$ for some non-decomposable $f \in F[x]$ and K has no nontrivial λ-quotients except F itself.

ii. K is an algebraic extension of F of finite degree if there is a sequence of simple extensions $F \subset K_1 \subset ... \subset K_s \subset K$.

iii. K is an algebraic extension of F if every $k \in K$ is contained in a finite degree algebraic extension L_k of F, which is contained in K.

We say that L is an algebraic closure of K, $L = \bar{K}$, if L is an algebraic extension of K, for any finite degree algebraic extension $K \hookrightarrow L_1$ we have a factorization $K \hookrightarrow L_1 \hookrightarrow L$ and L is minimal with respect to this property. We care only about monic f's because otherwise $F_1[x]/(f)$ contains torsion elements which are always nilpotent in any λ-structure. Some explanation why we have to accept that a field has a nontrivial quotient is contained in Remark 3.5.

Proposition 3.4: $\bar{F}_1 = F_1[\mathbf{Q}/\mathbf{Z}]$.

Proof. Observe first that the simple algebraic extension of F_1 is isomorphic to $F_1[x]/(x^p - 1)$ for some prime number p. This follows directly from 3.1 because if our extension is of positive degree and has presentation as $F_1[x]/(f)$ then 3.1 implies that $x^k - 1$ divides f for a certain positive k. But then either our extension has the forbidden quotient $F_1[x]/(x^k - 1)$ (we will say in the future that f has a forbidden factor) or $f = x^p - 1$ for a prime p. So we see that every simple algebraic extension of F_1 is isomorphic to $F_1[\mu_p]$ for a prime number p which can also be interpreted as a group ring $\mathbf{Z}[C_p]$ for a cyclic group C_p of order p.

Now assume that K is a simple algebraic extension of $F_1[\mu_n]$. We know that K has a presentation as $F_1[\mu_n][x]/(f)$ for a certain non-decomposable $f \in F_1[\mu_n][x]$. Observe, that for any $a \in F_1[\mu_n]$, $\psi^n(a) \in \mathbf{Z}$. Moreover for any $g \in F_1[\mu_n][x]$, $\psi^n(g) \in F_1[x]$ by the description of Adams operations. We are using here first of all the fact that the $F_1[\mu_n][x] = F_1[\mu_n] \otimes_{\mathbf{Z}} F_1[x]$ and the λ-operations on the tensor product are induced from those on the factors. Secondly, Adams operations are fully described by λ-operations and they are natural ring homomorphisms.

By the discussion above we know that $\psi^n(f) \in F_1[x]$. We can apply to it the same procedure as in the proof of 3.1 and get that if a is a root of $\psi^n(f)$ then all its powers have also this property. This implies that any root of $\psi^n(f)$ is a root of 1 or 0 and the statement of 3.1 is fulfilled for $\psi^n(f)$. Because $f|\psi^n(f)$ we know that all roots of f are roots of 1 of degree not exceeding nk, where $k = deg(f)$ (zero can be excluded from our considerations). The polynomial f cannot have any root of 1 of degree n. If this would be the case then, because $x^n - 1$ is fully decomposable in $F_1[\mu_n][x]$, f would have a forbidden quotient. Consider first the special case and assume that f has a root of prime degree p, and $(n, p) = 1$. This implies that $x^p - 1|\psi^n(f)$. But $f|\psi^n(f)$ and f and $x^p - 1$ have a common root. Moreover $x^p - 1$ has no root besides 1 in $F_1[\mu_n]$. Hence $x^p - 1|f$ and this implies that $f = x^p - 1$ or f has forbidden quotient. Now we can consider the general case. Let a be a root of f. We know that $a^{nk} = 1$ and $a^n \neq 1$. This implies that a is a root of 1 of degree ns and $s|k$. We have $a^{ns} = (a^s)^n = 1$ and hence a^s is a root of 1 of degree n. By the same considerations as in the special case we get now that $f = x^p - \mu_n^i \in F_1[\mu_n][x]$ and $p = s$ is a prime.

Observe that with n and p as in the special case we have

$$F_1[\mu_n][x]/(x^p - 1) = \mathbf{Z}[C_n][C_p] = \mathbf{Z}[C_n \times C_p] = \mathbf{Z}[C_{np}] = F_1[\mu_{np}]$$

In the general case observe that $a \cdot \mu_n$ is of degree pn and freely generates $F_1[\mu_n][x]/(x^p - \mu_n^i)$ over $\mathbf{Z}$ so we can write:

$$F_1[\mu_n][x]/(x^p - \mu_n^i) = \mathbf{Z}[C_{np}] = F_1[\mu_{np}]$$

This implies directly our proposition.

Remark 3.5: Unlike F_1 all field extensions of F_1 have nontrivial quotients so they contain "ideals". Obviously $(x - 1)|(x^n - 1)$ and this implies that there is a λ-ring map $F_1[\mu_n] \to F_1$ given by augmentation. This is caused by the fact that we induce our structures from the category M_{ab} where one point monoid $\mathbf{1}$ is a zero object. It means that beside the map $\mathbf{1} \to M$ for any $M \in M_{ab}$ we have also $M \to \mathbf{1}$ and this should be reflected in the category of λ-rings. Moreover in M_{ab} we have a notion of a quotient object, the quotient map there commutes always with the action of $\mathbf{N}^*$ so we have to allow the existence of quotients of our "fields".

Since we have the algebraic closure of F_1 we can look for the "powers of the Frobenius morphism". Recall the well known fact from algebraic geometry over finite fields. If $\bar{F}_p$ is an algebraic closure of F_p and X is a variety over F_p then

$$X(F_{p^n}) = X(\bar{F}_p)^{\mathbf{n}}$$

Here $X(K)$ denotes the K-points of X and $Y^{\mathbf{n}}$ denotes the fixed points of the nth power of the Frobenius morphism action on Y. We would like to have the similar formula over F_1. For the points of the variety over different fields we have universal solution. If R is a λ-ring and X is a variety over F_1 then

$$X(R) = Mor(R, X)$$

where the morphism set is taken in the category of varieties. In the case when X is described by another λ-ring S ($X = spec(S)$) we have as usual

$$Mor(R, X) = Hom_{\lambda-rings}(S, R)$$

A λ-ring structure on R implies that there is the λ-ring homomorphism $\lambda_R : R \to W(R)$, where $W(R)$ is a ring of big Witt vectors over R. This homomorphism is defined by lambda operations on R. More precisely, let $\Lambda(R)$ denote the universal λ-ring of R , see [Y, chapter 2] for the definition and properties. As a set it is equal to the set of formal power series over R with constant term 1 and addition defined as multiplication of power series. Then $\lambda_R = E \circ \lambda_t$, where λ_t was defined in Section 2 and E is the Artin-Hasse exponential isomorphism of $\Lambda(R)$ and $W(R)$ (for precise definitions and properties see [Y, chapter 4]). For any n we have a Frobenius morphism $f_n : W(R) \to W(R)$, which is the best approximation in the characteristic 0 to the Frobenius morphism known from finite characteristics ($f_p(a) = a^p mod(p)$). If we wiew $W(R)$ as invertible formal power series in indeterminant t with addition given by series multiplication and multiplication coming from ghost coordinates then $\lambda_R(r) = 1 + rt$ for any $r \in R$ of degree one. For such elements

$$f_n(1 + rt) = (1 + r^n t) = \psi^n(1 + rt)$$

Hence we have good reasons to claim that Frobenius action of $n \in N$ on $\mathbf{Z}[M]$ is realized by rising monoidal generators to the nth power. In other words it means that this action is realized by the Adams operations.

Let now $X = \mathbf{Z}[x]$. Then $X(\bar{F}_1) = Hom_{\lambda-rings}(\mathbf{Z}[x], F_1[\mathbf{Q}/\mathbf{Z}])$. Observe that every λ-homomorphism from $\mathbf{Z}[x]$ is determined by the image of x which should be contained in elements of degree not exceeding 1. In the case of monoidal rings for monoids with the cancelation law we calculate

$$Hom_{\lambda-rings}(\mathbf{Z}[x], \mathbf{Z}[M]) = M_+$$

because as an image of x can be taken any element of M or 0 and by 2.2 that is all. As usual $M_+ = M \cup \{0\}$. The Frobenius action is realized by Adams operations. Hence we calculate that the set $X(\bar{F_1})^{\mathbf{n}}$ consists of the roots of unity of degree $n-1$ plus additional element 0 so has cardinality n. From this we get the following formula for the ordinary Riemann ζ-function ζ_R:

$$\zeta_R(s) = 1 \ + \ \sum_{n=2}^{\infty} 1/|X(\bar{F_1})^{\mathbf{n}}|^s$$

where X is an affine line over F_1 equal to $\mathbf{Z}[x]$.

IV. Categorical ζ function over F_1.

Let us start from recalling after Kurokawa (compare [K]) the definition of the zeta function of a category with 0. If $\mathcal{C}$ is a category with 0 we say that $X \in Ob(\mathcal{C})$ is simple if for any object Y the set $Hom_{\mathcal{C}}(X, Y)$ consists only of monomorphisms and 0. Let $N(X)$, the norm of X, denote the cardinality of the set $End_{\mathcal{C}}(X)$. We say that an object X is finite if $N(X)$ is finite. We denote by $P(\mathcal{C})$ the isomorphism classes of all non zero finite simple objects of $\mathcal{C}$. Then we define the zeta function of $\mathcal{C}$ as

$$\zeta(s, \mathcal{C}) = \prod_{P \in P(\mathcal{C})} (1 - N(P)^{-s})^{-1}$$

In [K] Kurokawa studied the properties of such zeta functions but for us the crucial is:

Remark 4.1: Let Ab denote the category of abelian groups and ζ_R stands for the Riemann zeta function of the integers. Then

$$\zeta(s, Ab) = \zeta_R$$

The following observation was the starting point for our considerations and it underlines the role of the category of abelian monoids. In [Be] we proved:

Theorem 4.2:

$$\zeta_R \cdot (1 - 2^{-s})^{-1} = \zeta(s, M_{ab})$$

Hence the category of monoids carries all the information needed for calculating the Riemann ζ-function of the integers. We want to look at its categorical calculation from 4.1 in a slightly different way, which is suitable for generalizations. First of all we underline that while being in Ab we are working in the category of $\mathbf{Z}$-modules. There are good analogs of the category of modules over an object X of an abstract category $\mathcal{C}$ which has

0 and all finite limits. Beck in [Bec] defined them as abelian group objects in the category of objects over X (see also [H, chapter 2]). As is shown in [H] the category of abelian group objects in the category of rings over a given ring R is equivalent to the category of R-modules, where an R-module X defines the square zero extension of R with X as a square-zero ideal. In the case of $R = \mathbf{Z}$ we get, as expected, the category of abelian groups. An abelian group X corresponds to the square zero extension $\mathbf{Z} \triangleright X$. The finite simple objects in the category of rings over $\mathbf{Z}$ are easily seen to come from the simple abelian groups (finite cyclic groups C_p of prime order p). For a given p we see that $N(C_p)$ is equal to the cardinality of the set $Hom_{Rings/\mathbf{Z}}(\mathbf{Z}[x], \mathbf{Z} \triangleright C_p)$. The polynomial ring $\mathbf{Z}[x]$ is treated as a ring over $\mathbf{Z}$ via the map which takes x to 0. Observe that this agrees with our leading idea that new structures should be induced from the maps in the category of monoids, this time from the map $\mathbf{N}^+ \to \mathbf{1}$. The considerations above imply that we have the geometrical method for calculating the categorical ζ-function of integers. We just have to count the $\mathbf{Z} \triangleright C_p$-points of the affine line over $\mathbf{Z}$.

We can perform the calculation as above for any commutative ring R because finite simple objects in the category of R-modules correspond to the maximal ideals $I \subset R$ with finite quotient and one checks immediately that the cardinality of $Hom_{Rings/R}(R[x], R \triangleright R/I)$ is the same as the cardinality of $Hom_{R-mod}(R/I, R/I)$. Moreover this cardinality is the same as the number of elements of the residue field at the closed point corresponding to I in $Spec(R)$. From this it is obvious that we are really calculating the classical ζ-function of an affine variety $Spec(R)$ which is the same as categorical ζ-function of the category of R-modules.

Observe that we can perform the same calculations in the category M_{ab}, where the role of integers is played by the field of one element in the sense of [D]. But this gives us no new insight because in the world of monoids the field of one element is represented by one point monoid $\mathbf{1}$ consisting of the unite only, so we have equality of categories $M_{ab}/\mathbf{1} = M_{ab}$. But of course, in the spirit of our previous statements, the calculation from 4.2 can be presented as counting the $\mathbf{1} \triangleright M$ points of the affine line $\mathbf{N}^+$ over $\mathbf{1}$.

Below we show that we get the Riemann ζ-function of the integers via counting the number of $F_1 \triangleright M$-points of $F_1[x]$ in the category of rings over F_1 where M runs through finite simple objects in the category of F_1-modules. We have to start from describing the latter category in some accessible way. We will follow closely [H] because in [H, chapter 2] this category is described in full details for any λ-ring. The constructions uses the functor W from unital commutative rings to λ-rings which takes any ring R to its ring of Witt vectors $W(R)$. Originally the functor W was defined for rings with multiplicative unit. But the universal polynomials which define addition, multiplication and opposite in $W(R)$ do not use multiplicative unit so using the same formulas one can define the value of W on non-unital rings.

Recall that if R is a λ-ring then it comes with the λ-ring map $\lambda_R : R \to W(R)$ which is defined by lambda operations on R. More precisely, if $\Lambda(R)$ denote the ring of invertible formal power series over R with the leading coefficients equal to 1 then $\lambda_R = E \circ \lambda_t$ where E is the Artin-Hasse exponential isomorphism of $\Lambda(R)$ and $W(R)$ (see [Y, chapter 4]) and

$$\lambda_t(r) = \sum_{i=0}^{\infty} \lambda^i(r) t^i$$

As it is proved in [H], the category of modules over a λ-ring R, which is equal to the category $(Ring^\lambda/R)^+$ of abelian group objects in $Ring^\lambda/R$, is equivalent to the category $R-mod^\lambda$ of λ-modules over R. A λ-module over R is an R module M with a map $\lambda_M : M \to W(M)$ which is equivariant with respect to the λ-structure of R. Here $W(X)$ denotes the Witt ring construction applied to the non-unital ring X with trivial multiplication. It is easy to check that in this case $W(M)$ has also trivial multiplication and additively is equal to the infinite product of M. It is shown in [H, Lemma 2.2] that we have an isomorphism of rings

$$i : W(R) \triangleright W(M) \to W(R \triangleright M)$$

which is induced by the canonical inclusions of R and M into $R \triangleright M$. A λ-module M corresponds in the equivalence of $(Ring^\lambda/R)^+$ and $R-mod^\lambda$ to the λ-ring $R \triangleright M$ with the λ-ring structure defined by the composition

$$R \triangleright M \xrightarrow{\lambda_R \oplus \lambda_M} W(R) \triangleright W(M) \xrightarrow{i} W(R \triangleright M)$$

We have another description of the category $R - mod^\lambda$ (see [H, Remark 2.6]). If M is an object of this category and $\lambda_M : M \to W(M)$ is a structural map then it has components $\lambda_{M,n} : M \to M$ because as sets $W(M) = \prod_N M$. Easy calculation shows that $\lambda_{M,n}$ is $\psi_{R,n}$ equivariant, where $\psi_{R,n}$ is the nth Adams operation of R. This gives us description of the category $R - mod^\lambda$ as a category of left modules over a twisted monoid algebra $R^\psi[\mathbf{N}^*]$ where the multiplicative monoid $\mathbf{N}^*$ acts on any object M through the maps $\lambda_{M,n}$.

With the understanding of the category $R-mod^\lambda$ presented above we can come back to our situation and analyze the category $F_1 - mod^\lambda$. Observe that the Newton formula which relates Adams and λ-operations gives $\psi_{F_1,n} = id$ for any natural n. This implies that $\lambda_{M,1} = id$ and for $n > 1$, $\lambda_{M,n} : M \to M$ is any (additive) group homomorphism.

Lemma 4.3: Every object (M, λ_M) in $F_1 - mod^\lambda$ consists of an abelian group M and a sequence of group homomorphisms $\lambda_{M,n} : M \to M$ satisfying $\lambda_{M,n} \circ \lambda_{M,m} = \lambda_{M,mn}$ and $\lambda_{M,1} = id$. Morphisms $(M, \lambda_M) \to (P, \lambda_P)$ are given as group homomorphisms $f : M \to P$ which satisfy $f \circ \lambda_{M,n} = \lambda_{P,n} \circ f$ for any natural n.

The description of $F_1 - mod^\lambda$ was achieved before the statement of the lemma. But let us make here one comment. Our category of F_1-modules is the same as the category of modules over $\mathbf{Z}[\mathbf{N}^*]$ which is almost the same as the monoid algebra over $\mathbf{Z}$ of the multiplicative monoid of integers. Hence we did not leave the old approach to the field of one element, presented for example in [D], but it seems that we are getting more subtle methods of approaching the Riemann ζ-function.

Lemma 4.4: Assume M is a finite simple object in $F_1 - mod^\lambda$. If the set

$$Hom_{Rings/F_1}(\mathbf{Z}[x], \mathbf{Z} \triangleright M)$$

has finite cardinality different from 0 then M is of the form (C_p, λ_{C_p}), where C_p is the cyclic group of prime order p and $\lambda_{C_p,n} = id$ for $n \geq 1$.

Proof. Assume

$$1 < n(\mathbf{Z} \triangleright M) = |Hom_{Rings/F_1}(\mathbf{Z}[x], \mathbf{Z} \triangleright M)| < \infty.$$

The structure of the semi-direct product implies that if $\varphi \in Hom_{Rings/F_1}(\mathbf{Z}[x], \mathbf{Z} \triangleright M)$ then $\varphi(x) = (0, m)$ for a certain $m \in M$. Because φ is a λ-ring homomorphism it has to commute with λ-operations on the source and the target. Recall that in $\mathbf{Z}[x]$, $\lambda^n(x) = 0$ for $n > 1$. Hence for $n > 1$ we calculate

$$0 = \varphi(\lambda^n(x)) = \lambda^n(\varphi(x)) = \lambda^n((0, m)).$$

It means that we are looking for such objects (M, λ_M) and $m \in M$ which give us vanishing of higher λ-operations on elements $(0, m) \in \mathbf{Z} \triangleright M$. Now we can use the general formula for the Artin-Hasse isomorphism [Y, Section 4.2]. For any ring R if $f(t) = 1 + \Sigma a_i t^i \in \Lambda(R)$ then we write $f(t) = \prod(1 - (-1)^i b_i t^i)$ and the Artin-Hasse isomorphism $E : \Lambda(R) \to W(R)$ takes f to the sequence $(b_1, b_2, b_3, ...)$. This implies immediately that if λ-operations of degree bigger than 1 act trivially on $r \in R$ then this element has trivial (above the first) Witt coordinates in $W(R)$.

Now we come back to our considerations. The map $\lambda_M : M \to W(M)$ takes $m \in M$ to $(\lambda_{M,1}(m), \lambda_{M,2}(m)), \lambda_{M,3}(m), ...)$. Assume that $\lambda_{\mathbf{Z} \triangleright M}((0, m)) = ((a_1, y_1), (a_2, y_2), ...)$. It is obvious that $a_n = 0$ for $n \geq 1$. Artin-Hasse isomorphism described above tells us that $y_1 = m$ and $y_n = 0$ for $n > 1$. On the other hand calculations from Addendum 2.3 of [H] tell us that

$$\lambda_{M,n}(m) = y_1 + y_n.$$

Hence $\lambda_{M,n}(m) = m$ as we wanted to show.

The set $M' = \{m \in M \mid \lambda_{M,n}(m) = m \; for \; n = 1, 2, ...\}$ is a subgroup of M preserved by the λ-operations so it is a subobject of M in $F_1 - mod^\lambda$. But M was supposed to be simple what implies that $M' = M$. The λ-operations act trivially on M so M has to be simple as an abelian group. This observation implies our lemma.

Corollary 4.5: Recall that ζ_R denotes the Riemann ζ-function of integers. Let $\mathcal{C} = F_1 - mod^\lambda$ and $n(M) = |Hom_{Rings/F_1}(\mathbf{Z}[x], \mathbf{Z} \triangleright M)|$. We have

$$\zeta_R(s) = \prod_{M \in P'(\mathcal{C})} (1 - n(M)^{-s})^{-1}$$

where $P'(\mathcal{C})$ denotes nontrivial classes in $P(\mathcal{C})$.

FINAL REMARK. It seems that the success of the Deligne's approach to Weil conjectures was caused by the fact that for an affine variety over a finite field we have two ways of calculations the ζ function. One is classical (categorical) via looking at finite simple objects in corresponding category. The second is via algebraic closure, rich geometric structure and calculation of the same function via counting fixed points of the Frobenius action. In the present paper we tried to justify the statement that in the category of λ-rings the same two approaches should work.

V. Bibliography:

[Bec] - J. M. Beck: *Triples, algebras and cohomology.* Reprints in Theory and Applications of Categories 2 (2003) 1-59.
[Be] - S. Betley: *Riemann zeta via the category of monoids.* Kodai Math. Journal 36 (3) (2013) 487-490.
[B] - J. Borger: *Λ-rings and the field with one element.* Preprint, 2009.
[BS] - J. Borger, B. de Smit: *Galois theory and integral models of Λ-rings.* Bull. London Math. Soc. 40 (2008) 439-446.
[C] - A. Connes, C. Consani, M. Marcolli: *Fun with F_1.* J. Number Theory 129 (2009) 1532-1561.
[D] - A. Deitmar: *Schemes over F_1.* Progr. Math 239 (2005) 87-100.
[H] - L. Hesselholt: *The big de Rham-Witt complex.* Acta Math 214 (2015) 135-207.
[K] - N.Kurokawa: *Zeta Functions of Categories.* Proc. Japan Acad. 72 (1996) 221-222.
[KOW] - N.Kurokawa, H. Ochiai, M.Wakayama: *Absolute derivations and zeta functions.* Doc. Math., Extra Volume for Kazuya Kato's 50 Birthday, (2003) 565-584.
[S] - C.Soule: *Les variétés sur le corps a un élément.* Mosc. Math. J. 4 (2004), 217–244.
[Y] - D. Yau: *Lambda-Rings.* World Scientific 2010.

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