The Calabi Conjecture

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These are notes to the talk given on 9th March 2012 at the Graduate Topology and Geometry Seminar at the University of Warsaw. They are based almost entirely on [H] and [J], where a much more competent and detailed exposition can be found.

Fundamentals of Kähler geometry

Let (M, J) be a complex manifold with a complex structure J. We call M a *Kähler manifold* if it is equipped with a Riemannian metric g such that:

1. g is an *Hermitian metric*, i. e. it is compatible with the complex structure:

$$g(v, w) = g(Jv, Jw).$$

2. The induced real 2-form ω defined by $\omega(v, w) = g(Jv, w)$ is closed, i. e. $\mathrm{d}\omega = 0$.

In this case g is called a *Kähler metric* and ω is called the *Kähler form of g*.

Decomposition of the metric tensor. The complex structure J gives rise to a decomposition of the complexified tangent bundle $TM \otimes \mathbb{C}$ into two subbundles $T^{(1,0)}M$ and $T^{(0,1)}M$. Namely, for a point $p \in M$,

$$T_pM \otimes \mathbb{C} = T_p^{(1,0)}M \oplus T_p^{(0,1)}M,$$

where $T_p^{(1,0)}M, T_p^{(0,1)}M$ are the eigenspaces of J_p in $T_pM \otimes \mathbb{C}$ with eigenvalues i and $-i$, respectively (these are the only eigenvalues of J_p since $J_p^2 = -I$). In holmorphic coordinates $z_1, \dots, z_n$ the first one is spanned by $\partial/\partial z_1, \dots, \partial/\partial z_n$ and the second one by $\partial/\partial \bar{z_1}, \dots, \partial/\partial \bar{z_n}$.

This gives us a similar decomposition of the complexified cotangent bundle

$$TM^* \otimes \mathbb{C} = T^{*(1,0)}M \oplus T^{*(0,1)}M$$

and, consequently, a decomposition of every complex tensor bundle into subbundles coming from the decompositions of $TM \otimes \mathbb{C}$ and $T^*M \otimes \mathbb{C}$.

In particular, the metric tensor g_{ab} can be split into four pieces:

$$g_{ab} = g_{\alpha\beta} + g_{\bar{\alpha}\beta} + g_{\alpha\bar{\beta}} + g_{\bar{\alpha}\bar{\beta}}$$

Here, the Greek and overline indices denote tensors which are components of the decomposition of g_{ab}, that is $g_{\alpha\beta} \in \otimes^2 T^{*(1,0)}M$, $g_{\bar{\alpha}\beta} \in T^{*(0,1)}M \otimes T^{*(1,0)}M$ etc.

By direct calculations it can easily be shown that g is Hermitian if and only if

$$g_{\alpha\beta} = g_{\bar{\alpha}\bar{\beta}} = 0, \quad \text{i. e.}$$

$$g_{ab} = g_{\alpha\bar{\beta}} + g_{\bar{\alpha}\beta}.$$

Moreover, in that case the matrix $g_{\alpha\bar{\beta}}$ is Hermitian, that is, $\overline{g_{\alpha\bar{\beta}}} = g_{\beta\bar{\alpha}}$. It follows that $g_{\alpha\bar{\beta}}$ has only real eigenvalues and $\det(g_{\alpha\bar{\beta}})$ is a real function on M.

The Ricci form. Let g be a Kähler metric on M, ∇ – the Levi-Civita connection associated with g and $R = R^a_{bcd}$ – the Riemann curvature tensor, that is:

$$R(u, v) = \nabla_u \nabla_v - \nabla_v \nabla_u - \nabla_{[u,v]}.$$

For each pair (u, v) of tangent vectors the Riemann curvature defines a linear map $w \mapsto R(w, u)v$ from the tangent space to itself. We define the *Ricci curvature tensor* as the trace of this endomorphism:

$$\mathrm{Ric}(u, v) = \mathrm{tr}(R(\cdot, u)v).$$

In the index notation it is just a contraction of the curvature tensor: $R_{bd} = R^a_{bad}$. The Ricci tensor is a symmetric tensor of the type $(0, 2)$. Moreover, thanks to the J-invariance of the metric g it is also J-invariant, i. e. $\mathrm{Ric}(u, v) = \mathrm{Ric}(Ju, Jv)$. Thus we can use the complex structure J to produce a 2-form in the same manner as we get a Kähler form from an Hermitian metric. This real 2-form is denoted by ρ and called, not surprisingly, the *Ricci form*:

$$\rho(u, v) = \mathrm{Ric}(Ju, v).$$

A Kähler metric is called *Ricci-flat* when its Ricci form (or, equivalently, its Ricci tensor) vanishes identically.

Theorem

The Ricci form ρ is closed and its cohomology class $[\rho]$ in $H^2(M, \mathbb{R})$ is equal to $2\pi c_1(M)$, where $c_1(M)$ is the first Chern class of M. Therefore, it depends only on the complex structure on M.

Moreover, in holomorphic coordinates we can express ρ explicitly as a differential:

$$\rho = -\frac{1}{2}\mathrm{dd}^c \left(\log \det(g_{\alpha\bar{\beta}})\right).$$

We will use this expression later to reformulate the Calabi conjecture.

The holonomy group. Let (M, g) be a compact Riemannian manifold. Denote by $\mathrm{Hol}(g)$ its *holonomy group*, that is, the subgroup of automorphisms of the tangent space in a given point $p \in M$ induced by the parallel transport around all closed loops based at p. Berger's theorem classifies all possible holonomy groups of compact Riemannian manifolds which are simply-connected, irreducible (not locally a product of Riemannian manifolds) and nonsymmetric (not locally a symmetric space). Among them, groups $\mathrm{U}(n), \mathrm{SU}(n)$ and $\mathrm{Sp}(n) \subset \mathrm{SU}(2n)$ play an important role, as they are inseparably connected with Kähler geometry. Namely, under the aforementioned conditions, we have

Theorem
$\mathrm{Hol}(g) \subseteq \mathrm{U}(n)$ *if and only if* g *is a Kähler metric. If* g *is a Kähler metric, then* $\mathrm{Hol}(g) \subseteq \mathrm{SU}(n)$ *if and only if* g *is Ricci-flat.*

The proof is based on the so-called *holonomy principle* which establishes a bijective correspondence between the $\mathrm{Hol}_p(g)$-invariant tensors on T_pM in a given point $p \in M$ and the parallel (with respect to the Levi-Civita connection) tensor fields on M. As both $\mathrm{U}(n)$ and $\mathrm{SU}(n)$ can be described as subgroups of $\mathrm{SO}(2n)$ preserving certain tensors (the complex structure and the holomorphic volume form on $\mathbb{C}^n = \mathbb{R}^{2n}$, respectively), the embeddings of $\mathrm{Hol}_p(g)$ into $\mathrm{U}(n)$ and $\mathrm{SU}(n)$ give rise to globally defined, parallel tensor fields: a complex structure J and a holomorphic volume form Ω. The existence of such tensor fields is equivalent for a metric to be, respectively, Kähler and Ricci-flat.

There is, however, some work to do, as, for example, J is *a priori* only an almost complex structure and one has to check that it is actually a complex structure (that is, it satisfies a certain integrability condition). Details may be found in [H].

The Calabi conjecture

We can now formulate the famous conjecture which was posed by Eugenio Calabi in 1954 and eventually proved by Shing-Tung Yau twenty years later.

The Calabi conjecture
Let M *be a compact, complex manifold, and* g *a Kähler metric on* M *with Kähler form* ω*. Then for each real, closed* $(1,1)$*-form* ρ' *on* M *such that* $[\rho'] = 2\pi c_1(M)$ *in* $H^2(M, \mathbb{R})$ *there exists a unique Kähler metric* g' *on* M *with Kähler form* ω'*, such that* $[\omega] = [\omega']$ *in* $H^2(M, \mathbb{R})$ *and the Ricci form of* g' *is* ρ'*.*

In the specific case of $c_1(M) = 0$ we can take $\rho' = 0$ and obtain the following theorem:

Theorem
Let M *be a compact Kähler manifold with vanishing first Chern class. Then there exists a Ricci-flat Kähler metric on* M*. Every such a metric is uniquely determined by the cohomology class of its Kähler form.*

This theorem shows the importance of the Calabi conjecture. Berger's list describes all *possible* holonomy groups, whereas the result of Calabi and Yau proves that there actually exist compact manifolds with holonomy groups $\mathrm{SU}(n), \mathrm{Sp}(n)$. They are called, respectively, *Calabi-Yau* and *hyperkähler manifolds* and they are of great significance in mathematics and theoretical physics, as they arise naturally in geometry and string theory. There are in fact plenty of them and the theorem gives us an easy way to recognise them: to decide wheter a given complex manifold admits a Calabi-Yau or hyperkähler structure we just have to check if its first Chern class vanishes. In many cases it is possible.

Reformulating the conjecture. One of the key step in proving the Calabi conjecture was an observation by Calabi that it can be restated as an equivalent problem of the existence and uniqueness of the smooth solution of a certain partial differential equation. Subsequently, Yau used various methods of analysis to solve the problem. Before that, some progress had been made by Calabi and Aubin.

Assume that Kähler metric g' satisfies the thesis of the Calabi conjecture, that is $[\omega] = [\omega']$ and g' has Ricci form ρ'. We will find a partial differential equation for g'.

We will need the following important theorem from Kähler geometry.

$\mathrm{dd^c}$-Lemma

Let M be a compact Kähler manifold. Then a closed form η on M is exact if and only if it is $\mathrm{dd^c}$-exact, that is, of the form $\eta = \mathrm{dd^c}\xi$ for some form ξ.

The next proposition follows immediately from the $\mathrm{dd^c}$-Lemma and the fact that the kernel of $\mathrm{dd^c}$ consists precisely of constant functions (for the proofs see [Ba]).

Lemma

Let M be a compact, complex manifold and let g, g' be Kähler metrics with Kähler forms ω, ω', respectively. If ω and ω' have the same cohomology class in $H^2(M, \mathbb{R})$, then there exists a smooth, real function ϕ on M such that

$$\omega' = \omega + \mathrm{dd^c}\phi.$$

Such a function is unique up to the addition of a constant.

In particular, as $[\omega] = [\omega']$, there exists a smooth real function ϕ on M such that

$$\omega' = \omega + \mathrm{dd^c}\phi.$$

Because ϕ is unique up to the addition of a constant, we may specify it uniquely by adding the condition

$$\int_M \phi \ \mathrm{d}V_g = 0,$$

where $\mathrm{d}V_g$ is the Riemannian volume form induced by g. As ϕ specifies ω', and ω', if it is positive, specifies the Hermitian metric g', we would like to find a partial differential equation satisfied by ϕ. We have already used the condition $[\omega] = [\omega']$, so now we should make use of the fact that the Ricci form of g' is ρ'.

To do this, we will need some equations connecting the Kähler form, the Ricci form and the metric. Denote by n the complex dimension of M. Then the n-th exterior power $\omega^n = \omega \wedge \ldots \wedge \omega$ of the symplectic form is proportional to the Riemannian volume form:

$$\omega^n = n!\ \mathrm{d}V_g,$$

or, in local holomorphic coordinates $z_1, \ldots, z_n$,

$$\omega^n = i^n n! \det(g_{\alpha\bar{\beta}})\ \mathrm{d}z_1 \wedge \mathrm{d}\bar{z}_1 \wedge \ldots \wedge \mathrm{d}z_n \wedge \mathrm{d}\bar{z}_n. \tag{1}$$

As regards the Ricci form, we have already mentioned the following local expression:

$$\rho = -\frac{1}{2}\mathrm{dd}^c \left(\log\det(g_{\alpha\bar{\beta}})\right). \tag{2}$$

Now, as previously, from $[\rho] = [\rho'] = 2\pi c_1(M)$ we deduce the existence of a real smooth function f on M, unique up to the addition of a constant, such that

$$\rho' = \rho - \frac{1}{2}\mathrm{dd}^{\mathrm{c}} f.$$

Define a smooth, positive function F on M by the condition $(\omega')^n = F\omega^n$. From equation (1) for ω and ω' we can see that $F = \det(g'_{\alpha\bar{\beta}})/\det(g_{\alpha\bar{\beta}})$ and using equation (2) we get

$$\frac{1}{2}\mathrm{dd}^{\mathrm{c}}(\log F) = \frac{1}{2}\mathrm{dd}^{\mathrm{c}}\left(\log\det(g'_{\alpha\bar{\beta}}) - \log\det(g_{\alpha\bar{\beta}})\right) = \rho - \rho' = \frac{1}{2}\mathrm{dd}^{\mathrm{c}} f.$$

Thus $\mathrm{dd}^{\mathrm{c}}(f - \log F) = 0$ and it follows that the function $f - \log F$ is constant on M (once more we use the fact that $\ker \mathrm{dd}^{\mathrm{c}}$ consists precisely of constant functions). Define a constant $A > 0$ by $f - \log F = -\log A$. Then $F = Ae^f$ and we obtain an equation for ϕ:

$$(\omega + \mathrm{dd}^{\mathrm{c}}\phi)^n = (\omega')^n = Ae^f\omega^n,$$

where f is given (because ρ' is given) and the constant A can be easily expressed using f. Since $[\omega'] = [\omega]$, by equation (1) and Stokes' theorem we have

$$A\int_M e^f\ \mathrm{d}V_g = \frac{1}{n!}\int_M (\omega')^n = \frac{1}{n!}\int_M \omega^n = \mathrm{vol}_g(M).$$

Therefore, as this reasoning can be easily reversed, we have found the following equivalent version of the Calabi conjecture.

The Calabi Conjecture

Let M be a compact, complex manifold of the dimension n, and g a Kähler metric on M with the Kähler form ω. Let f be a smooth real function on M and define $A > 0$ by $A \int_M e^f \, \mathrm{d}V_g = \mathrm{vol}_g(M)$. Then there exists a unique smooth real function ϕ such that

(a) $\omega + \mathrm{dd^c}\phi$ *is a positive* $(1,1)$*-form,*

(b) $\int_M \phi \, \mathrm{d}V_g = 0$,

(c) $(\omega + \mathrm{dd^c}\phi)^n = Ae^f \omega^n$.

Moreover, in local holomorphic coordinates $z_1, \ldots, z_n$ condition (c) can be expressed in the following way:

$$\det \left(g_{\alpha\bar{\beta}} + \frac{\partial^2 \phi}{\partial z_\alpha \partial \bar{z}_{\bar{\beta}}} \right) = Ae^f \det \left(g_{\alpha\bar{\beta}} \right). \tag{3}$$

Remark

Condition (a) actually follows from condition (c) (see [J]).

Notice that equation (3) is an elliptic, second-order partial differential equation of a kind known as a *Monge-Ampére equation*. Therefore we have reformulated the purely geometric Calabi conjecture into the equivalent problem from analysis. The theory of elliptic partial differential equations is very rich and well-developed and there are many sophisticated methods for proving the existence, uniqueness and regularity of solutions for such equations. The main problem, however, is that equation (3) is highly nonlinear, as it is nonlinear in the derivatives of the highest order. This is the difficulty of the Calabi conjecture and the reason it remained unsolved for so many years.

Outline of the proof

A detailed exposition of the proof can be found in [J]. The idea is as follows.

The continuity method. We want to prove that our equation

$$(\omega + \mathrm{dd^c}\phi)^n = Ae^f \omega^n$$

has a solution ϕ for every f. However, in the special case of $f = 0$ we get the equation

$$(\omega + \mathrm{dd^c}\phi)^n = \omega^n,$$

which certainly has a solution $\phi = 0$. The idea, which is called the *continuity method*, is to consider a one-parameter family of equations

$$(\omega + \mathrm{d}d^c\phi)^n = A_t e^{tf} \omega^n$$

connecting our original equation with the equation which has a solution. Then we consider a set $S \subseteq [0,1]$ of these $t \in [0,1]$ for which there exists a solution ϕ_t of the corresponding equation. Of course $0 \in S$, so if we are able to prove that S is both closed and open in $[0,1]$, the theorem is proved as S is nonempty and the interval is connected.

Auxiliary theorems. Everywhere M is supposed to be a compact, complex manifold and g is a Kähler metric on M with the Kähler form ω.

For $\alpha \in (0,1)$ and an integer $k \geq 0$ we denote by $C^{k,\alpha}$ the Banach space of k-differentiable functions on M with α-Hölder continuous k-th derivatives. Since the definition of the norm must be global, we cannot use coordinates and expressions like $\partial f/\partial z_k$, hence the precise definition is somewhat complicated.

We call the following equations

$$\int_M \phi \, \mathrm{d}V_g = 0, \quad (\omega + \mathrm{dd^c}\phi)^n = Ae^f\omega^n$$

the *Calabi equations for* (f, A).

Theorem C1 (a priori estimates)
Let $Q \geq 0$. Then there exists a constant $P \geq 0$ depending only on M, g and Q such that the following holds.

Suppose $f \in C^3$, $A > 0$ and $\phi \in C^5$ satisfies the Calabi equations for (f, A). If $\|f\|_{C^3} \leq Q$, then

$$\|\phi\|_{C^0} \leq P, \quad \|\mathrm{dd^c}\phi\|_{C^0} \leq P \quad \text{and} \quad \|\nabla\mathrm{dd^c}\phi\|_{C^0} \leq P.$$

Theorem C2 (regularity)
Let $Q \geq 0$ and $\alpha \in (0,1)$. Then there exists a constant $P \geq 0$ depending only on M, g, Q and α such that the following holds.

Suppose $f \in C^3$, $A > 0$ and $\phi \in C^5$ satisfies the Calabi equations for (f, A). If

$$\|f\|_{C^{3,\alpha}} \leq Q, \quad \|\phi\|_{C^0} \leq Q, \quad \|\mathrm{dd^c}\phi\|_{C^0} \leq Q \quad \text{and} \quad \|\nabla\mathrm{dd^c}\phi\|_{C^0} \leq Q$$

then $\phi \in C^{5,\alpha}$ and $\|\phi\|_{C^{5,\alpha}} \leq P$. Moreover, if $f \in C^{k,\alpha}$ for $k \geq 3$ then $\phi \in C^{k+2,\alpha}$, and if $f \in C^\infty$ then $\phi \in C^\infty$.

Theorem C3 (openess)
Fix $\alpha \in (0,1)$ and suppose that $f_0 \in C^{3,\alpha}, A_0 > 0$ and $\phi_0 \in C^{5,\alpha}$ satisfies the Calabi equations for (f_0, A_0).

Then whenever $f \in C^{3,\alpha}$ and $\|f - f_0\|_{C^{3,\alpha}}$ is sufficiently small, there exist $A > 0$ and $\phi \in C^{5,\alpha}$ which satisfies the Calabi equations for (f, A).

Theorem C4 (uniqueness)
Let $f \in C^1$ and $A > 0$. Then there is at most one function $\phi \in C^3$ which satisfies the Calabi equations for (f, A).

The proof of the Calabi conjecture. In the situation of the previous paragraph, fix $\alpha \in (0,1)$ and define S to be the set of all numbers $t \in [0,1]$ for which there exist $A > 0$ and $\phi \in C^{5,\alpha}$ satisfying

$$\int_M \phi \, \mathrm{d}V_g = 0, \quad (\omega + \mathrm{dd^c}\phi)^n = Ae^{tf}\omega^n.$$

We will show that S is both closed and open in $[0,1]$. For that we need the following well-known result of the Rellich-Kondrashov type.

Theorem
Let (M,g) be a Riemannian manifold, $k \in \mathbb{N}$ and $\alpha \in (0,1)$. The natural embedding $C^{k,\alpha}(M) \hookrightarrow C^k(M)$ is compact, that is every bounded subset of $C^{k,\alpha}(M)$ is relatively compact in $C^k(M)$.

Take a sequence of numbers t_i from S convergent to $t \in [0,1]$. By the definition of S there exist $A_i > 0$ and $\phi_i \in C^{5,\alpha}$ satisfying the Calabi equations for $(t_i f, A_i)$. Define $Q = \|f\|_{C^{3,\alpha}}$. Then for all i we have $t_i \in [0,1]$ and $\|t_i f\|_{C^{3,\alpha}} \leq Q$, hence we can apply Theorem C1 to obtain the existence of P such that

$$\|\phi_i\|_{C^0} \leq P, \quad \|\mathrm{dd^c}\phi_i\|_{C^0} \leq P \quad \text{and} \quad \|\nabla \mathrm{dd^c}\phi_i\|_{C^0} \leq P$$

for all i. By Theorem C2, there exists $R > 0$ such that $\|\phi_i\|_{C^{5,\alpha}} \leq R$. The sequence (ϕ_i) is therefore bounded in $C^{5,\alpha}$ and consequently, by the previous theorem, it has a subsequence convergent in C^5. Take a subsequence of functions ϕ_i in such a way that it converges and the corresponding subsequence of numbers A_i also converges. Denote the limit of the former by ϕ, and of the latter by A. It can easily be seen that ϕ satisfies the Calabi equations for (tf, A). Theorems C1 and C2 guarantee that $\phi \in C^{5,\alpha}$, hence the set S contains its limit points, and is closed.

Openess of S follows immediately from Theorem C3. As S is nonempty (because, obviously, $0 \in S$) and $[0,1]$ is connected, S must be the whole interval. In particular, $1 \in S$, so there exists $\phi \in C^{5,\alpha}$ satisfying the Calabi equations for (f, A). Theorem C2 shows that ϕ is actually smooth, and Theorem C4 that it is unique. This ends the proof of the Calabi conjecture.

References

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