

Equivariant Chern classes and localization theorem

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Abstract

For a complex variety with a torus action we propose a new method of computing Chern-Schwartz-MacPherson classes. The method does not apply resolution of singularities. It is based on Localization Theorem in equivariant cohomology.

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Equivariant cohomology is a powerful tool for studying complex manifolds equipped with a torus action. The Localization Theorem of Atiyah and Bott and the resulting formula of Berline-Vergne allow to compute global invariants of singular subsets in terms of the fixed points of the action. We will concentrate on the Chern-Schwartz-MacPherson classes. The global class is determined by the local contributions coming from the fixed points. On the other hand, the sum of the local Chern classes divided by the Euler classes is equal to zero in an appropriate localization of equivariant cohomology. Especially for Grassmannians we obtain interesting formulas with nontrivial relations involving rational functions. We discuss the issue of positivity: the local Chern class may be presented in a various ways, depending on the choice of generating circles of the torus. For some choices we find that the coefficients of the presentation are nonnegative. Also the coefficients in the Schur basis are nonnegative in many examples, but it turns out that not always.

In §6 we propose a new method of computing equivariant Chern-Schwartz-MacPherson classes which does not apply resolution of singularities. Our main example is the determinant variety, the subset of square matrices defined by the equation $\det = 0$. We study its compactification, the Schubert

variety of codimension one in $\text{Grass}_n(\mathbb{C}^{2n})$. We discuss computational difficulties appearing for that example. The concrete formula for the Chern class is a huge sum of fractions. Surprisingly all the difficulties lie in simplifying that expression.

1 Equivariant fundamental class

Let M be a complex manifold and $X \subset M$ a closed complex subvariety. The *fundamental class* of X , which is the Poincaré dual of the cycle defined by X is denoted by

$$[X] \in H^{2 \text{codim}(X)}(M).$$

When the ambient manifold M is contractible, for example when M is an affine space, there is no use of $[X]$ since the cohomology of M is trivial. An interesting situation appears when an algebraic group G acts on M and X is preserved by the action. In that case there is an equivariant fundamental class of X which belongs to the equivariant cohomology of M

$$[X] \in H_G^{2 \text{codim}(X)}(M).$$

Now even if M is contractible we obtain a remarkable invariant of the pair (M, X) . For contractible M its equivariant cohomology coincides with the equivariant cohomology of a point

$$H_G^{2 \text{codim}(X)}(M) \simeq H_G^{2 \text{codim}(X)}(pt)$$

and the cohomology of a point *is the ring of characteristic classes for G* . In particular

- if $G = (\mathbb{C}^*)^n$ then $H_G^*(pt) = \mathbb{Q}[t_1, t_2, \dots, t_n]$
- if $G = GL_n$ then $H_G^*(pt) = \mathbb{Q}[\sigma_1, \sigma_2, \dots, \sigma_n] = \mathbb{Q}[t_1, t_2, \dots, t_n]^{\Sigma_n}$
- in general the ring of characteristic classes coincides with the invariants of the Weyl group acting on the characteristic classes for the maximal torus

$$H_G^*(pt) = H_T^*(pt)^W.$$

(We consider here only cohomology with rational coefficients.)

An extended discussions of different names for the equivariant fundamental class can be found in [9, §2.1]. For a review of equivariant cohomology see e.g. [15].

For $G = GL_n$ equivariant cohomology and the equivariant fundamental classes $[X] \in H_{GL_n}^*(pt)$ has turned out to be an adequate tool for studying the Thom polynomials of singularities of maps. Here X is a set of singular jets in the space of all jets of maps. Its equivariant fundamental class $[X]$

is the universal characteristic class which describes cohomological properties of singular loci of maps. Last decade appeared a series of papers by Rimanyi and his collaborators (starting from [31]) and Kazarian (see e.g. [21]). Powerful tools allowing effective computations were developed and some structure theorems were stated. Geometric approach to equivariant cohomology leads to positivity results [29, 26, 27]. The source of these results is the following principle:

Theorem 1 *If $X \subset \mathbb{C}^N$ is a cone in a polynomial representation of GL_n , then $[X]$ is a nonnegative combination of Schur functions.*

The examples of polynomial representations are the following: the natural representation, its tensor products, symmetric products, exterior products and in general quotients of the sums of tensor products. The Schur functions constitute a basis of the ring of characteristic classes

$$H_{GL_n}^*(pt) = H^*(\text{Grass}_n(\mathbb{C}^\infty))$$

corresponding to the decomposition of the infinite Grassmannian into Schubert cells. For an algebraic treatment of Schur functions see [24].

A version of Theorem 1 holds for G being a product of the general linear groups. We will be interested in torus actions. Theorem 5 stated in [30] reduces to:

Theorem 2 *Let $T = (\mathbb{C}^*)^n$ and let $t_1, t_2, \dots, t_n \in \text{Hom}(T, \mathbb{C}^*)$ be the characters corresponding to the decomposition of T into the product. Suppose $V = \bigoplus V_\lambda$ is a representation of T such that each weight λ appearing in V is a nonnegative combination of t_i 's. Let $X \subset V$ be a variety preserved by T -action. Then the equivariant fundamental class $[X] \in H_T^*(V) = \mathbb{Q}[t_1, t_2, \dots, t_n]$ is a polynomial with nonnegative coefficients.*

2 Equivariant Chern class

Our goal is to study more delicate invariants of the subvarieties in representations. In the most of interesting cases the subvarieties to study are singular. Our first choice is the equivariant version of the Chern-Schwartz-MacPherson classes. We recall that the usual Chern-Schwartz-MacPherson classes [25] live in homology, they are Poincaré duals of the Chern classes of the tangent bundle when the variety is smooth. These classes are functorial in a certain sense, and therefore usually they are computed via resolution of singularities. The equivariant version of Chern-Schwartz-MacPherson classes was developed by Ohmoto [28].

Definition 3 *For an equivariant subvariety X in a G -manifold M let $c^G(X)$ denote the image of the equivariant Chern-Schwartz-MacPherson class in the equivariant cohomology of the ambient space $H_G^*(M)$.*

Remark 4 By its nature Chern-Schwartz-MacPherson class lies in equivariant homology of X , but we apply Poinré duality in the ambient manifold M .

Let us see what our invariant means for conical sets in the affine space.

Example 5 Let $T = \mathbb{C}^*$ acts on \mathbb{C}^n by scalar multiplication. As in [4] consider an affine cone $X \subset \mathbb{C}^n$ and compute its Chern-Schwartz-MacPherson class in \mathbb{P}^n . Let $h = c_1(\mathcal{O}(1)) \in H^2(\mathbb{P}^n)$. We claim that if

$$c(X) = a_0 + a_1 h + \cdots + a_n h^n \in H^*(\mathbb{P}^n),$$

then

$$c^T(X) = a_0 + a_1 t + \cdots + a_n t^n \in H_T^*(\mathbb{C}^n),$$

where $t \in H_T^2(pt)$ is the identity character. Thus $c^T(X)$ coincides with the invariant of conical sets studied by Aluffi-Marcolli. Moreover:

- $a_n = 1$, since $\chi(X) = 1$,
- $a_0 + a_1 h + \cdots + a_{n-1} h^{n-1} = c(\mathbb{P}(X)) \in H^*(\mathbb{P}^{n-1})$.

The proof is given in §10. This is a reflection of *Localization Theorem* which will be discussed in §3.

Now suppose $G = T = (\mathbb{C}^*)^n$ and let T acts on a vector space V as in Theorem 2. We pose a question:

Question. *When $c^T(X) \in H_T^*(V) = \mathbb{Q}[t_1, t_2, \dots, t_n]$ has nonnegative coefficients?*

This is a special property of X since in general the answer is negative. For example suppose $T = \mathbb{C}^*$ acts diagonally on $V = \mathbb{C}^N$ with weight 1. Let X be a cone over a curve of genus > 1 . Then $c^T(X) = [X] + 2(1-g)t^{N-1} + t^N$ is negative at the gradation $2(N-1)$. On the other hand we have a bunch of positive examples: local Chern classes have nonnegative coefficients for

- toric singularities (see Corollary 9),
- generic hyperplane arrangements with a small number of hyperplanes [2],
- banana Feynman motives [3].

3 Localization theorem

For the moment we leave the question of positivity. Our current goal is to develop a calculus which would allow to compute equivariant Chern classes avoiding resolution of singularities. Our main tool is the Localization Theorem for torus action. The topological setup is the following: suppose the torus $T = (S^1)^n$ or $(\mathbb{C}^*)^n$ acts on a compact space M (decent enough, e.g. equivariant CW-complex). The equivariant cohomology $H_T^*(M)$ is a module over equivariant cohomology of the point

$$H_T^*(pt) = \mathbb{Q}[t_1, t_2, \dots, t_n].$$

Theorem 6 (Atiyah–Bott, [6]) *The restriction to the fixed set*

$$\iota^* : H_T^*(M) \longrightarrow H_T^*(M^T)$$

becomes an isomorphism after localizing in the maximal ideal

$$\mathfrak{m} = (t_1, t_2, \dots, t_n) \subset \mathbb{Q}[t_1, t_2, \dots, t_n].$$

If M is a manifold, then the inverse of the restriction map is given by the Berline-Vergne formula. To explain that let us fix a notation. We decompose the fixed point set into components $M^T = \bigsqcup_{\alpha \in A} M_\alpha$. Each M_α is a manifold and denote by $e_\alpha \in H_T^*(M_\alpha) = H^*(M_\alpha) \otimes \mathbb{Q}[t_1, t_2, \dots, t_n]$ the equivariant Euler class of the normal bundle. The following map is the inverse of the restriction to the fixed points

$$\begin{aligned} H_T^*(M^T)_{\mathfrak{m}} &= \bigoplus_{\alpha \in A} H_T^*(M_\alpha)_{\mathfrak{m}} \xrightarrow{\simeq} H_T^*(M)_{\mathfrak{m}} \\ \{x_\alpha\}_{\alpha \in A} &\mapsto \sum_{\alpha \in A} \iota_{\alpha*} \left(\frac{x_\alpha}{e_\alpha} \right), \end{aligned} \tag{1}$$

where $\iota_\alpha : M_\alpha \hookrightarrow M$ is the inclusion. The key point in the Berline-Vergne formula is that the Euler class e_α is *invertible* in $H_T^*(M_\alpha)_{\mathfrak{m}}$.

Furthermore consider the push-forward, i.e. the integration along M

$$p_* = \int_M : H_T^*(M) \rightarrow H_T^{*-2\dim(M)}(pt)$$

where $p : M \rightarrow pt$ is the constant map. Another form of the localization theorem allows to express the integration along M by integrations along components of the fixed point set.

Theorem 7 ([10]) *For $x \in H_T^*(M)$ the integral can be computed by summation of local contributions*

$$\int_M x = \sum_{\alpha \in A} \int_{M_\alpha} \frac{x|_{M_\alpha}}{e_\alpha}. \tag{2}$$

In particular, when the fixed point set is discrete $M^T = \{p_0, p_1, \dots, p_n\}$ then the Euler class is the product of weights

$$e_p = \prod_{\lambda \in \Lambda} \lambda^{\dim(V_\lambda)} \in H_T^*(\{p\}) \in \mathbb{Q}[t_1, t_2, \dots, t_n],$$

provided that $\mathbb{T}_p M$, the tangent space at p is the sum of weight spaces

$$\mathbb{T}_p M = \bigoplus_{\lambda \in \Lambda} V_\lambda.$$

The integral along M is equal to the sum of fractions:

$$\int_M a = \sum_{p \in M^T} \frac{a|_p}{e_p}.$$

Remark 8 The Berline-Vergne formula (2) can be formulated for singular spaces embedded into a smooth manifold. The local factor $\frac{1}{e_\alpha}$ is replaced by $\frac{[X]_{M_\alpha}}{e_\alpha}$, see [13, 9]. There is a generalization of the formula (1) for equivariant homology (or Chow groups) of singular spaces, but one needs an additional assumption allowing to define ι^* , [13, Proposition 6].

4 Toric varieties

We keep in mind that our purpose is computation of equivariant Chern classes. From Localization Theorem it follows that equivariant Chern classes are determined by local Chern classes belonging to the cohomologies of the components of M^T . In the beginning let us consider the toric varieties, which are quite easy, but unfortunately not very general from our point of view.

Let $X = M$ be a smooth toric variety. Consider the cycle Ξ_X which is equal to the sum of the closures of orbits. We will show that Ξ_X represents the equivariant Chern class of X .

- First note that if $X = \mathbb{C}^1$ with the standard action of $T = \mathbb{C}^*$ then indeed the Chern class is equal $[\mathbb{C}] + [0] = \Xi_X$.
- By Whitney formula and the product property of sets the statement holds for $X = \mathbb{C}^n$ with the standard action of $T = (\mathbb{C}^*)^n$.
- Every smooth toric variety locally looks like \mathbb{C}^n with the standard action of the torus, therefore the equation $c^T(X) = \Xi_X$ holds locally, i.e. after restriction to each fixed point.
- Let X be a complete smooth toric variety. Then $H_T^*(X)$ is free over $H_T^*(pt)$. By the Localization Theorem $c^T(X) = \Xi_X$ holds globally.

- The noncomplete case follows since any smooth toric variety can be compactified equivariantly.
- The singular case can be deduced as usual by functoriality. One argues that the Chern class of the constructible function supported by a single orbit is exactly the fundamental class of the closure of that orbit without boundary cycles.

The non-equivariant case was proven by Ehlers and Barthel-Brasselet-Fieseler [7] and it also follows immediately from [1].

We note that the cycle representing the equivariant Chern class is effective. Therefore by Theorem 2 we have:

Corollary 9 *Let V be a representation of $T = (\mathbb{C}^*)^n$. Suppose an affine toric variety (possibly singular) is embedded equivariantly into V . If the weights of the torus acting on V are nonnegative then the coefficients of $c^T(X) \in H_T^*(V) = \mathbb{Q}[t_1, t_2, \dots, t_n]$ are nonnegative.*

The situation described in the Corollary 9 appears when

$$X = X_\sigma = \text{Spec}(\mathbb{C}(\sigma^\vee \cap N))$$

is presented in the usual way: the embedding into

$$V = \text{Spec}(\mathbb{C}[x_1, x_2, \dots, x_n])$$

is given by a choice of the generators of the semigroup $\sigma^\vee \cap N$, see [14, §1.3].

5 Some calculi of rational functions

Before examining equivariant Chern classes of Schubert varieties let us look closer at some computations based on Localization Theorem for Grassmannians. Let us start with the projective space $M = \mathbb{P}^n$ with the standard torus $T = (\mathbb{C}^*)^{n+1}$ action. The fixed point set is discrete and consists of coordinate lines

$$M^T = \{p_0, p_1, \dots, p_n\}.$$

The tangent space at the point p_k decomposes into one dimensional representations:

$$\mathbb{T}_{p_k} M = \bigoplus_{\ell \neq k} \mathbb{C}_{t_\ell - t_k}.$$

The Euler class is equal to

$$e_{p_k} = \prod_{\ell \neq k} (t_\ell - t_k).$$

Let us integrate powers of $c_1 := c_1(\mathcal{O}(1))$. Of course

$$\int_{\mathbb{P}^n} c_1^m = \begin{cases} 0 & \text{for } m < n \\ 1 & \text{for } m = n \end{cases}$$

Applying Berline-Vergne formula we get the identity

$$\sum_{k=0}^n \frac{(-t_k)^m}{\prod_{\ell \neq k} (t_\ell - t_k)} = \begin{cases} 0 & \text{for } m < n \\ 1 & \text{for } m = n, \end{cases} \quad (3)$$

which is not obvious at the first sight. For example try to compute by hand the sum

$$\frac{t_0^2}{(t_1-t_0)(t_2-t_0)(t_3-t_0)} + \frac{t_1^2}{(t_0-t_1)(t_2-t_1)(t_3-t_1)} + \frac{t_2^2}{(t_0-t_2)(t_1-t_2)(t_3-t_2)} + \frac{t_3^2}{(t_0-t_3)(t_1-t_3)(t_2-t_3)}.$$

This is exactly the expression (3) for $m = 2$, $n = 3$. Replacing

$$t_0 = 0, \quad t_1 = 1, \quad t_2 = 2, \quad \dots, \quad t_n = n$$

(i.e. specializing to a subtorus) the sum (3) is equal to

$$\sum_{k=0}^n \frac{(-1)^{m+k} k^m}{k!(n-k)!}$$

Multiplying by $n!$ we obtain

$$\sum_{k=0}^n \binom{n}{k} (-1)^{m+k} k^m = \begin{cases} 0 & \text{for } m < n \\ n! & \text{for } m = n. \end{cases}$$

which is a good exercise for students.

The integral of higher powers of c_1 is even more interesting: Let us see what do we get for $m > n$? For example $n = 2$, $m = 4$ we have

$$\frac{t_0^4}{(t_1-t_0)(t_2-t_0)} + \frac{t_1^4}{(t_0-t_1)(t_2-t_1)} + \frac{t_2^4}{(t_0-t_2)(t_1-t_2)}$$

It takes some time to check that the sum is equal to

$$t_0^2 + t_1^2 + t_2^2 + t_0 t_1 + t_0 t_2 + t_1 t_2, .$$

In terms of the elementary symmetric functions it is equal to

$$\sigma_1^2 - \sigma_2 .$$

Proposition 10 *In general*

$$\int_{\mathbb{P}^n} c_1^{n+k}$$

is equal to the Schur function S_k (which corresponds to the Segre class of vector bundles).

Proof. By Jacobi-Trudy formula

$$S_k(t_0, t_1, \dots, t_n) = \frac{\begin{vmatrix} t_0^{n+k} & t_0^{n-1} & t_0^{n-2} & \dots & t_0^1 & 1 \\ t_1^{n+k} & t_1^{n-1} & t_1^{n-2} & \dots & t_1^1 & 1 \\ t_2^{n+k} & t_2^{n-1} & t_2^{n-2} & \dots & t_2^1 & 1 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ t_n^{n+k} & t_n^{n-1} & t_n^{n-2} & \dots & t_n^1 & 1 \end{vmatrix}}{\prod_{i < j} (t_i - t_j)}.$$

To prove the proposition it is enough to use Laplace expansion with respect to the first column and watch carefully the signs. \square

We will have a look now at the calculus on $\text{Grass}_m(\mathbb{C}^n)$. The fixed point set consists of coordinate subspaces:

$$\text{Grass}_m(\mathbb{C}^n)^T = \{p_\lambda : \lambda = (\lambda_1 < \lambda_2 < \dots < \lambda_m), 1 \leq \lambda_1, \lambda_m \leq n\}$$

The tangent spaces at the fixed point p_λ decomposes into distinct line representations of T :

$$\mathbb{T}_{p_\lambda} \text{Grass}_m(\mathbb{C}^n) = \bigoplus_{k \in \lambda, \ell \notin \lambda} \mathbb{C}_{t_\ell - t_k}.$$

The Euler class is equal to

$$e_{p_\lambda} = \prod_{k \in \lambda, \ell \notin \lambda} (t_\ell - t_k).$$

Let us integrate a characteristic class of the tautological bundle \mathcal{R}_m . Suppose that the class $\phi(\mathcal{R}_m)$ is given by a symmetric polynomial in Chern roots $W(x_1, x_2, \dots, x_m)$. Then

$$\int_{\text{Grass}_m(\mathbb{C}^n)} \phi(\mathcal{R}_m) = \sum_{\lambda} \frac{W(t_i : i \in \lambda)}{\prod_{k \in \lambda, \ell \notin \lambda} (t_\ell - t_k)}$$

It looks like a rational function, but we obtain a polynomial in $\{t_i \mid i = 1, 2, \dots, m+n\}$ of degree $\deg(W) - \dim(\text{Grass}_m(\mathbb{C}^n))$. According to [8] this expression is equal to the iterated residue

$$\text{Res}_{z_1=\infty} \text{Res}_{z_2=\infty} \dots \text{Res}_{z_m=\infty} \frac{W(z_1, z_2, \dots, z_m) \prod_{i \neq j} (z_i - z_j)}{\prod_{i=1}^n \prod_{j=1}^m (t_i - z_j)}. \quad (4)$$

Of course if $\deg(W) < \dim(\text{Grass}_m(\mathbb{C}^n)) = (n-m)m$, then

$$\sum_{\lambda} \frac{W(t_i : i \in \lambda)}{\prod_{k \in \lambda, \ell \notin \lambda} (t_\ell - t_k)} = 0$$

If $\deg(W) = \dim(\text{Grass}_m(\mathbb{C}^n))$, then we get a constant. For example for $W = c_1^{\dim(\text{Grass}_m(\mathbb{C}^n))} = [-(x_1 + x_2 + \dots + x_m)]^{(n-m)m}$ we obtain the degree of the Plücker embedding $\text{Grass}_m(\mathbb{C}^n) \subset \mathbb{P}(S^m(\mathbb{C}^n))$ (or the volume of $\text{Grass}_m(\mathbb{C}^n)$). According to Hook Formula [16, §4.3]

$$\deg(\text{Grass}_m(\mathbb{C}^n)) = \frac{(m(n-m))!}{\prod_{(i,j) \in \lambda} h(i,j)},$$

where $h(i,j)$ denotes the length of a hook with vertex at $(i,j) \in \lambda$ contained in the rectangle $m \times (n-m)$. For $\text{Grass}_3(\mathbb{C}^7)$ the hook lengths are the following

6	5	4	3
5	4	3	2
4	3	2	1

Hence the degree is equal to

$$\frac{12!}{6 \cdot 5 \cdot 4 \cdot 3 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 4 \cdot 3 \cdot 2 \cdot 1} = 462.$$

It would be interesting to find an immediate connection of the Hook formula and the residue method given by the formula (4).

Let us now formulate a generalization of Proposition 10. For a partition $I = (i_1 \geq i_2 \geq \dots \geq i_n)$ the Schur function is defined by Jacobi-Trudy formula [24, §I.3]

$$S_I(t_1, t_2, \dots, t_n) = \frac{\begin{vmatrix} t_1^{n-1+i_1} & t_1^{n-2+i_2} & t_1^{n-3+i_3} & \dots & t_1^{1+i_{n-1}} & t_1^{i_n} \\ t_2^{n-1+i_1} & t_2^{n-2+i_2} & t_2^{n-3+i_3} & \dots & t_2^{1+i_{n-1}} & t_2^{i_n} \\ t_3^{n-1+i_1} & t_3^{n-2+i_2} & t_3^{n-3+i_3} & \dots & t_3^{1+i_{n-1}} & t_3^{i_n} \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ t_n^{n-1+i_1} & t_n^{n-2+i_2} & t_n^{n-3+i_3} & \dots & t_n^{1+i_{n-1}} & t_n^{i_n} \end{vmatrix}}{\prod_{1 \leq i < j \leq n} (t_i - t_j)}.$$

The definition of Schur function is extended to characteristic classes of vector bundles. Expanding the determinant with respect to the first block column containing $m \times m$ minors we find the formula for push-forward:

Theorem 11 *Consider the quotient bundle \mathcal{Q} and the tautological bundle \mathcal{R} over $\text{Grass}_m(\mathbb{C}^n)$. Let $J = (j_1 \geq j_2 \geq \dots \geq j_{n-m})$ and $K = (k_1 \geq k_2 \geq \dots \geq k_m)$ be partitions. Suppose $j_{n-m} - m \geq k_1$. Then*

$$\int_{\text{Grass}_m(\mathbb{C}^n)} S_J(\mathcal{Q}) S_K(\mathcal{R}) = S_I(t_1, t_2, \dots, t_n),$$

where $I = (j_1 - m \geq j_2 - m \geq \dots \geq j_{n-m} - m \geq k_1 \geq k_2 \geq \dots \geq k_m)$.

A suitable modifications of Theorem 11 can be easily formulated for the partitions not satisfying the inequality $j_{n-m} - m \geq k_1$. The integral is equal up a sign to the Schur function for another partition or it is zero. By splitting principle Theorem 11 implies the corresponding statement for Grassmannian bundles over any base, not necessarily over the classifying space BT . This way we obtain a proof of the Gysin homomorphism formula [20, Predlozhenie 1].

The equivariant Schubert calculus was studied by a number of authors: Knutson–Tao [22], Laksov–Thorup [23], Gatto–Santiago [17] and others. Some formulas can be obtained by taking residue at infinity [8, 9]. Concluding this section I would like to say that it seems that still the calculus of *rational* symmetric functions is not developed enough. In §7 we will present a method of computation of equivariant Chern classes of Schubert varieties. Unfortunately I do not know (maybe except Theorem 11) a tool which would allow to simplify the expressions which appear in computation.

6 Computing equivariant Chern classes without resolution of singularities

Below we sketch a method of computing the Chern class of a T -invariant singular variety not using a resolution of singularities.

Assume that the fixed point set of an action of the torus on a complex manifold M is discrete. For a given class $x \in H_T^k(M)$ of degree $k < 2 \dim(M)$ the integral $\int_M x$ vanishes. By the Localization Theorem also the sum $\sum_{p \in M^T} \frac{x|_p}{e_p}$ has to vanish. In particular if $x = c^T(X)$, then except from the top gradation

$$\sum_{p \in M^T} \frac{c^T(X)|_p}{e_p} = 0. \quad (5)$$

This relation between local equivariant Chern classes allows to compute them inductively. Suppose $M^T = \{p_0, p_1, \dots, p_N\}$ and assume that we know all local equivariant Chern classes for p_1, p_2, \dots, p_N . Then

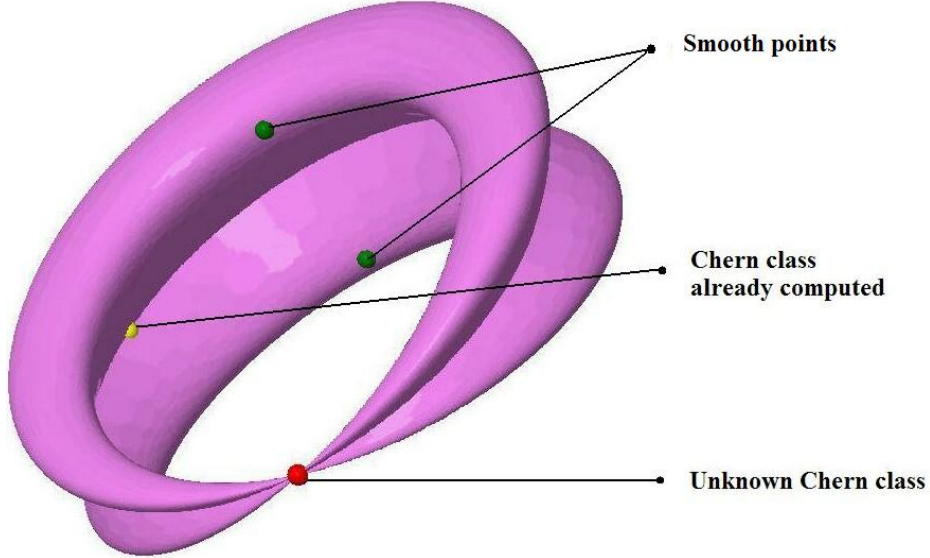
$$c^T(X_{p_0}) = - \sum_{i=1}^N \frac{e_{p_0}}{e_{p_i}} c^T(X_{p_i}) \quad (6)$$

except from the top gradation. For Grassmannians the quotient $\frac{e_{p_0}}{e_{p_i}}$ simplifies remarkably.

The top component of the local Chern class is easy. If $p \in X^T$ then the top Chern class is equal to the Euler class at the point p

$$c_{top}^T(X_p) = e_p \in H_T^{2 \dim(M)}(pt). \quad (7)$$

In fact this statement is the crucial point for computation. Any other equivariant characteristic class satisfies the relation 6. The condition fixing the top equivariant Chern class and vanishing for the degrees higher than the dimension of the ambient space makes the equivariant Chern class unique. We will prove the formula (7) in §11.



Computation of the local equivariant Chern classes

7 Computation of local equivariant Chern class of the determinant variety

Let us compute the local equivariant Chern class of the Schubert variety of codimension one in $\text{Grass}_n(\mathbb{C}^{2n})$

$$\Omega_1(n) = \{W : W \cap \langle \varepsilon_1, \varepsilon_2, \dots, \varepsilon_n \rangle \neq 0\}.$$

We will apply the method sketched above. Let us concentrate on the most singular point $p_{1,2,\dots,n}$. Let us start with $n = 2$. The canonical neighborhood of the point $p_{1,2}$ in $\text{Grass}_2(\mathbb{C}^4)$ is identified with

$$\text{Hom}(\text{span}(\varepsilon_1, \varepsilon_2), \text{span}(\varepsilon_3, \varepsilon_4))$$

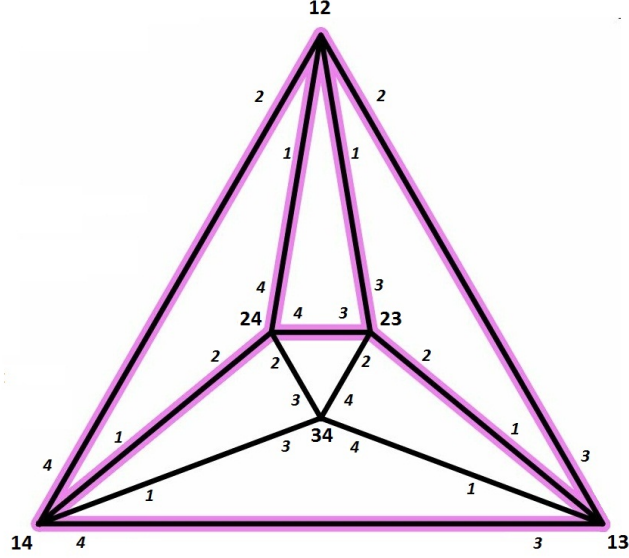
and the variety $\Omega_1(2)$ intersected with this neighbourhood consists of the singular linear maps $\mathbb{C}^2 \rightarrow \mathbb{C}^2$. The corresponding elements of $\Omega_1(2)$ are the planes spanned by the row-vectors of the matrix

$$\begin{pmatrix} 1 & 0 & a & b \\ 0 & 1 & c & d \end{pmatrix}.$$

The equation of $\Omega_1(2)$ is

$$\det \begin{pmatrix} a & b \\ c & d \end{pmatrix} = 0.$$

Before performing computations let us draw the Goresky-Kottwitz-MacPherson graph ([18, Th. 7.2]) for $M = \text{Grass}_2(\mathbb{C}^4)$ with the variety $\Omega_1(2)$ displayed.



Schubert variety Ω_1 in $\text{Grass}_2(\mathbb{C}^4)$.

The numbers attached to the edges indicate the weights of the T actions along the one dimensional orbits. For example at the point $p_{1,3}$ in the direction towards $p_{1,2}$ the action is by the character $t_2 - t_3$. The variety $\Omega_1(2)$ is singular at the point $p_{1,2}$ and it is smooth at the remaining points. For example at the point $p_{1,3}$ the coordinates are

$$\begin{pmatrix} 1 & a & 0 & b \\ 0 & c & 1 & d \end{pmatrix}$$

and the equation of $\Omega_1(2)$ is $b = 0$. For that point the local Chern class is equal to

$$(t_4 - t_1)(1 + t_2 - t_1)(1 + t_2 - t_3)(1 + t_4 - t_3).$$

The summand in the formula (5) is the following

$$\begin{aligned} & \frac{(t_4 - t_1)(1 + t_2 - t_1)(1 + t_2 - t_3)(1 + t_4 - t_3)}{(t_4 - t_1)(t_2 - t_1)(t_2 - t_3)(t_4 - t_3)} = \\ & = \left(1 + \frac{1}{t_2 - t_1}\right) \left(1 + \frac{1}{t_2 - t_3}\right) \left(1 + \frac{1}{t_4 - t_3}\right) \end{aligned}$$

We sum up the contribution coming from the fixed points $p_{1,3}$, $p_{1,4}$, $p_{2,3}$, $p_{2,4}$, simplify and multiply by $-(t_3 - t_1)(t_4 - t_1)(t_3 - t_2)(t_4 - t_2)$. We obtain

$$\begin{aligned}
(t_3 + t_4 - t_1 - t_2) & \quad \text{deg} = 1 \\
(t_3 + t_4 - t_1 - t_2)^2 & \quad \text{deg} = 2 \\
(t_3 + t_4 - t_1 - t_2)(2t_1t_2 - t_1t_3 - t_2t_3 - t_1t_4 - t_2t_4 + 2t_3t_4) & \quad \text{deg} = 3 \\
-4(t_3 - t_1)(t_4 - t_1)(t_3 - t_2)(t_4 - t_2) & \quad \text{deg} = 4
\end{aligned}$$

The terms of gradation < 4 coincide with the equivariant Chern class of $\Omega_1(2)$ localized at the point $p_{1,2}$. The result is symmetric in two groups of variables: $\{t_1, t_2\}$ and $\{t_3, t_4\}$. The coefficients of the expansion in the basis of the Schur functions

$$c^T(\Omega_1) = \sum a_{I,J} S_I(-t_1, -t_2) \cdot S_J(t_3, t_4)$$

has the following coefficients:

	0	1	11	2	21	22
0	1	1	1	2	1	
1	1	1	3	1	1	
11	1	3		1		
2	1	1	1			
21	2	1				
22	1					

Computations of the equivariant Chern class $\Omega_1(3) \subset \text{Grass}_3(\mathbb{C}^6)$ can be continued without problems by the same method. At the points of the type p_I with $|I \cap \{1, 2, 3\}| = 1$ the variety is smooth, while at the points p_I with $|I \cap \{1, 2, 3\}| = 2$ the singularity is of the type $\Omega_1(2)_{p_{1,2}} \times \mathbb{C}^5$. We write the sum of fractions according to the rule (6) and simplify. For example the expression which has to be simplified to compute the gradation one is the following:

$$\begin{aligned}
& - \frac{(s_3 - t_1)(s_3 - t_2)(s_1 - t_3)(s_2 - t_3)}{(s_3 - s_1)(s_3 - s_2)(t_1 - t_3)(t_2 - t_3)} (s_3 - t_3) + \text{sym.} + \\
& \frac{(s_3 - t_1)(s_3 - t_2)(s_1 - t_3)(s_2 - t_3)}{(s_1 - s_3)(s_2 - s_3)(t_3 - t_1)(t_3 - t_2)} (s_1 + s_2 - t_1 - t_2) + \text{sym.} \quad (8)
\end{aligned}$$

(Here $s_1 = t_4$, $s_2 = t_5$, $s_3 = t_6$.) The given summands are the contributions coming from the points $p_{3,4,5}$ and $p_{1,2,6}$. Of course the sum is equal to the fundamental class

$$[\Omega_1] = s_1 + s_2 + s_3 - t_1 - t_2 - t_3,$$

which may be computed in another way. This example shows how very complicated rational functions may in fact lead to a simple result. The

difficulty lies in simplifying that expression. Higher degree terms are much more complicated. We write the final result in the Schur basis

$$c^T(\Omega_1) = \sum a_{I,J} S_I(-t_1, -t_2, -t_3) \cdot S_J(s_1, s_2, s_3).$$

The coefficients are the following:

	0	1	11	2	111	21	3	211	31	22	311	221	32	321	222	33	331	322	332	333
0	1	2	2	4	5	1	9	3	4	6	9	3	8	4	1	3	6	3	1	
1	1	4	8	5	12	12	2	19	5	8	8	16	4	8	10	1	2	4	1	
11	2	8	12	9	16	16	3	18	6	8	6	10	4	4		1	1			
2	2	5	9	4	11	9	1	13	2	5	3	10	1	2	5			1		
111	4	12	16	11	8	16	4		6	10			4			1				
21	5	12	16	9	16	14	2	15	3	5	3	5	1	1						
3	1	2	3	1	4	2		3		1		2			1					
211	9	19	18	13		15	3		3	5			1							
31	3	5	6	2	6	3		3		1			1							
22	4	8	8	5	10	5	1	5	1		1									
311	6	8	6	3		3				1										
221	9	16	10	10		5	2		1											
32	3	4	4	1	4	1		1												
321	8	8	4	2		1														
222	4	10		5			1													
33	1	1	1		1															
331	3	2	1																	
322	6	4		1																
332	3	1																		
333	1																			

We note that all the coefficients are nonnegative.

While computing the equivariant Chern class of $\Omega_1(4) \subset \text{Grass}_4(\mathbb{C}^8)$ appears a problem with the size of the expressions, since $\dim(\text{Grass}_4(\mathbb{C}^8)) = 16$ and $\dim(T) = 8$. In a polynomial of degree 15 in 8 variables there are

490 314 monomials.

The expression is a sums of 68 fractions with factors $t_i - t_j$ in denominators. We might have used another compactification of \mathbb{C}^{16} , e.g. the projective space \mathbb{P}^{16} . There are less fixed points, but the denominators are more complicated. They are of the form $\prod[(t_i - t_j) - (t_k - t_\ell)]$.

One practical solution appears naturally. The fixed points can be divided into groups with $|I \cap \{1, 2, \dots, n\}|$ fixed. Let $f_k(u_\bullet, v_\bullet)$ be the expression for the local equivariant Chern class of $\Omega_1(k)$ with $u_\bullet = (t_1, t_2, \dots, t_k)$ and $v_\bullet = (t_{k+1}, t_{k+2}, \dots, t_{2k})$. The local equivariant Chern $c^T(\Omega_1(n))|_{p_1, 2, \dots, n}$ class can be computed by the formula (6), which becomes

$$-\sum_{k=1}^{n-1} \sum_{\substack{I \subset \{1, 2, \dots, 2n\} \\ |I| = n, I \cap \{1, 2, \dots, n\} = k}} \frac{e_{p_1, 2, \dots, n}}{e_{p_I}} f_k(I) g_k(I), \quad (9)$$

where $f_k(I)$ depends on the two group of variables

$$u_{\bullet} = t_{I \cap \{1,2,\dots,n\}} \quad \text{and} \quad v_{\bullet} = t_{\{n+1,n+2,\dots,2n\} \setminus I}$$

and $g_k(I)$ is the Chern class of the singular stratum of the type $\Omega_1(k)$. The factors in the quotients $\frac{e_{p_1,2,\dots,n}}{e_{p_I}}$ cancel out partially and miraculously all the summands for a fixed k turn out to be integral. For $n = 3$ and degree one the summands are given by the formula (8).

Such a division of fixed points has a geometric meaning. In fact we deal with the partial localization (see §10). Consider the action of the subtorus \mathbb{C}^* acting on \mathbb{C}^{2n} with weight 1 on the first n coordinates and with the weight -1 on the remaining coordinates. Then the fixed point set decomposes into disjoint union of the products of the Grassmannians:

$$\text{Grass}_n(\mathbb{C}^{2n})^{\mathbb{C}^*} = \bigsqcup_{k=0}^n \text{Grass}_k(\mathbb{C}^n) \times \text{Grass}_{n-k}(\mathbb{C}^n).$$

The summand for $k = 0$ consists of one point

$$\{0\} \oplus \langle \varepsilon_{n+1}, \varepsilon_{n+2}, \dots, \varepsilon_{2n} \rangle,$$

which does not belong to $\Omega_1(n)$, while for $k = n$ we have

$$\langle \varepsilon_1, \varepsilon_2, \dots, \varepsilon_n \rangle \oplus \{0\},$$

the point which we are concerned with. Let \mathcal{R}_k and \mathcal{Q}_k be the tautological and the quotient bundles over $\text{Grass}_k(\mathbb{C}^n)$. The result of the sum (9) is equal to

$$-(-1)^{n-k} \sum_{k=1}^{n-1} \int_{\text{Grass}_k(\mathbb{C}^n) \times \text{Grass}_{n-k}(\mathbb{C}^n)} [f_k(\mathcal{R}_k, \mathcal{Q}_{n-k}) \cdot \bar{g}_k],$$

where \bar{g}_k is (up to multiplication by a certain Euler class) the Chern class of the stratum of the singularity type $\Omega_1(k)$. Precisely

$$\begin{aligned} \bar{g}_k = e(\mathcal{Q}_k^* \otimes \mathcal{Q}_{n-k}) \cdot e(\mathcal{R}_k^* \otimes \mathcal{R}_{n-k}) \cdot \\ \cdot c(\mathcal{R}_k^* \otimes \mathcal{Q}_k) \cdot c(\mathcal{R}_{n-k}^* \otimes \mathcal{Q}_{n-k}) \cdot c(\mathcal{Q}_k \otimes \mathcal{R}_{n-k}^*). \end{aligned}$$

Using Fubini theorem we do not have to simplify a large expression in one step and we arrive to the result relatively quickly. Also knowing the Schur expansion of the functions f_k one can apply Theorem 11.

8 GKM-relations

Less time consuming method of the local equivariant Chern class is based on the relation discovered by Chen-Skjelbred [12], called GKM-relations

after the rediscovery in [18]. These relation allow to determine the local equivariant Chern class at the point p_I knowing only the local equivariant Chern classes at the neighbouring points in the GKM-graph. This is so since

$$c^T(V)|_{p_I} \equiv c^T(V)|_{p_J} \text{ modulo } (t_i - t_j),$$

whenever

$$J = (I - \{i\}) \cup \{j\}.$$

Again the method works for all the degrees smaller then the dimension of the Grassmannian, since the intersection of the ideals $(t_i - t_j)$ is contained in the gradation greater or equal to the dimension of the Grassmannian. That is so for any GKM-space. Now the problem of simplifying huge rational function is replaced by solving a relatively small system of linear equations.

9 The result for $\text{Grass}_4(\mathbb{C}^8)$

Let us write the local equivariant Chern class in the Schur basis

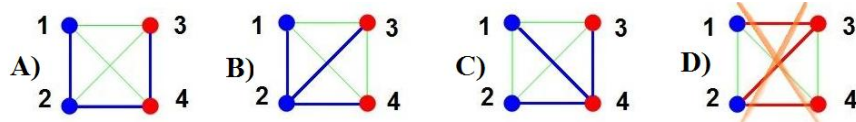
$$c^T(\Omega_1(4))|_{p_{1,2,3,4}} = \sum a_{I,J} S_I(-t_1, -t_2, -t_3, -t_4) \cdot S_J(t_5, t_6, t_7, t_8).$$

Just to quench readers curiosity we show here the most interesting fragment of the table of coefficients.

It is hard not to have impression that there should be a way of writing down this Chern class in a compact way. For example the Chern class of the tangent bundle written in the Schur basis is as much complicated as ours, but it is just $c(\text{Hom}(\mathcal{R}_n, \mathcal{Q}_n))$.

It turns out that the local equivariant Chern class of $\Omega_1(4)$ is a positive combination of monomials in $-t_1, -t_2, -t_3, -t_4, t_5, t_6, t_7, t_8$. As one can see it is *not* a positive combinations of products of Schur functions. Fortunately we do not have a contradiction with the conjecture of Aluffi and Mihalcea [5] which says that the Chern classes are effective. Note that the Schubert varieties are only T -invariant, and the Theorem 1 does not apply. Instead we have a freedom with choosing the basis of weights. The local Chern class is a polynomial in $u_{i,j} = t_i - t_j$. To write $c^T(X)$ in a unique way we chose a spanning tree of the full graph with vertices $1, 2, \dots, 2n$. The edge between i and j (with the orientation forced by the partition) corresponds to the generator $t_j - t_i$. Some choices lead to an expression with nonnegative coefficients.

Positive monomial bases for $\text{Grass}_2(\mathbb{C}^4)$



- A) $t_2 - t_1, \quad t_4 - t_2, \quad t_3 - t_4$
 B) $t_2 - t_1, \quad t_3 - t_2, \quad t_4 - t_2$
 C) $t_4 - t_1, \quad t_4 - t_2, \quad t_4 - t_3$
 D) $t_3 - t_1, \quad t_3 - t_2, \quad t_4 - t_2$ this is not a positive basis

The positivity condition for a graph is the following:

- Characters of the tangent representation are nonnegative sums of base elements.

That in fact supports the conjecture of Aluffi and Mihalcea in a stronger, equivariant version.

The original, nonequivariant version was checked by B. Jones [19] for cells in $\text{Grass}_m(\mathbb{C}^n)$ for $m \leq 3$. In his computations equivariant cohomology and Localization Theorem was used to compute the push-forward of classes from a resolution.

10 Partial localization

There exists the following modification of the localization formula: we can replace M^T by any invariant submanifold or even arbitrary invariant subset) Y containing the fixed point set M^T . Then the restriction map

$$H_T^*(M) \rightarrow H_T^*(Y)$$

becomes an isomorphism after the localization in the maximal ideal \mathfrak{m} . Also the Berline-Vergne formula holds, but it makes sense only for Y being a

submanifold. Suppose that $Y = Y_1 \sqcup \{p\}$. It follows that for any $x \in H_T^*(M)$ we have

$$\frac{x|_p}{e_p} + \int_{Y_1} \frac{x|_{Y_1}}{e_{Y_1}} = 0 \quad (10)$$

for gradation smaller than $\dim(M)$. We will apply this formula for lower gradations of $c^T(X)$. Also we need to know the integral of the top gradation. The integral of the top Chern-Schwartz-MacPherson class is the Euler characteristic and the same holds for the equivariant Chern class by the commutativity of the diagram:

$$\begin{array}{ccccc} c_{top}^T(X) & \in & H_T^{2\dim(M)}(M) & \longrightarrow & H^{2\dim(M)}(M) & \ni & c_{top}(X) \\ & & \downarrow & & \downarrow & & \\ \int_M c_{top}^T(X) & \in & H_T^0(pt) & \xrightarrow{\simeq} & H^0(pt) & \ni & \chi(X). \end{array}$$

We apply the partial localization and find that

$$\frac{c_{top}^T(X)|_p}{e_p} + \int_{Y_1} \frac{c_{top}^T(X)|_{Y_1}}{e_{Y_1}} = \chi(X). \quad (11)$$

Here e_{Y_1} is the equivariant Euler class of the normal bundle of Y_1 . (Of course it may be of different gradations over distinct components of Y_1 .)

Example 12 The partial localization allows to compute the Chern class of the affine cone over a projective variety. Suppose T acts on \mathbb{C}^n with nonzero weights

$$w_1, w_2, \dots, w_n.$$

First recall that the equivariant cohomology ring of \mathbb{P}^{n-1} is the quotient of the polynomial algebra

$$H_T^*(pt)[h] = \mathbb{Z}[t_1, t_2, \dots, t_n, h]$$

by the relation

$$\prod_{i=1}^n (h + w_i) = 0.$$

Using the elementary symmetric functions σ_i the relation takes form

$$\sum_{i=0}^n \sigma_i(w_\bullet) h^{n-i} = 0. \quad (12)$$

Let $X \subset \mathbb{C}^n$ be a T -invariant cone and $\mathbb{P}(X) \subset \mathbb{P}^{n-1}$ its projectivization. We consider X as a constructible set in \mathbb{P}^n and we will compute the equivariant

its Chern class. Denote by $\iota : \mathbb{P}^{n-1} \rightarrow \mathbb{P}^n$ the inclusion. The equivariant Chern class of X restricted to \mathbb{P}^{n-1} is equal to

$$\iota^*c(X) = \iota^*c(\overline{X}) - \iota^*\iota_*c(\mathbb{P}(X)) = (1+h) \cdot c(\mathbb{P}(X)) - h \cdot c(\mathbb{P}(X)) = c(\mathbb{P}(X)).$$

The equivariant Chern class of $\mathbb{P}(X)$ can be written as

$$c^T(\mathbb{P}(X)) = \sum_{i=0}^{n-1} a_i(t)h^i \in H_T^*(\mathbb{P}^{n-1})$$

for some polynomials $a_i(t) \in H_T^*(pt)$ of degree $\leq n-1-i$. To compute the local Chern class at 0 we will apply the formulas (10) and (11) to $M = \mathbb{P}^n$, $Y = \{0\} \cup \mathbb{P}^{n-1}$ and $Y_1 = \mathbb{P}^{n-1}$. We compute

$$\int_{\mathbb{P}^{n-1}} \frac{c^T(X)}{e_{Y_1}} = \int_{\mathbb{P}^{n-1}} \sum_{i=0}^{n-1} a_i(t)h^{i-1}.$$

Except from $i=0$ the summands are integral (belong to $H_T^*(\mathbb{P}^{n-1})$) and they are of the gradation smaller than $n-1$. Therefore

$$\int_{\mathbb{P}^{n-1}} \frac{c^T(X)}{e_{Y_1}} = \int_{\mathbb{P}^{n-1}} \frac{a_0(t)}{h}.$$

An easy calculation using (12) shows that the inverse Euler class of the normal bundle to \mathbb{P}^{n-1} is equal to

$$h^{-1} = - \sum_{i=1}^n \frac{\sigma_{n-i}(w_\bullet)}{\sigma_n(w_\bullet)} h^{i-1}.$$

Hence

$$\int_{\mathbb{P}^{n-1}} \frac{c^T(X)}{e_{Y_1}} = - \int_{\mathbb{P}^{n-1}} a_0(t) \sum_{i=1}^n \frac{\sigma_{n-i}(w_\bullet)}{\sigma_n(w_\bullet)} h^{i-1} = - \frac{a_0(t)}{\sigma_n(w_\bullet)}$$

By the formulas (10) and (11) and since $\sigma_n(w_\bullet) = e_p$ we find that

$$\frac{c^T(X)|_p}{e_p} - \frac{a_0(t)}{e_p} = \chi(X) = 1.$$

Corollary 13

$$c^T(X)|_p = a_0(t) + e_p. \tag{13}$$

The degree of a_0 is at most $n-1$, therefore it does not interfere with e_p , which is homogeneous of degree n .

Example 5 cont. From the general case we will derive the formula for the equivariant Chern class of Example 5. We assume that $T = \mathbb{C}^*$ and all the weights are equal to one. Assume that the usual, nonequivariant Chern class of $\mathbb{P}(X)$ is equal to

$$c(\mathbb{P}(X)) = \sum_{i=0}^{n-1} b_i x^i,$$

where $x = c_1(\mathcal{O}(1))$ and $b_i \in \mathbb{Z}$. The torus T acts on the fibers of the normal bundle to $Y_1 = \mathbb{P}^{n-1}$ with weights -1 , therefore the equivariant Euler class is equal to

$$h = x - t$$

under identification

$$H_T^*(\mathbb{P}^{n-1}) = H^*(\mathbb{P}^{n-1})[t].$$

For trivial T actions always the equivariant Chern class is equal to the usual one, hence

$$c^T(X) = \sum_{i=0}^{n-1} b_i (h + t)^i = \sum_{i=0}^{n-1} b_i t^i h^0 + \text{higher powers of } h.$$

Therefore by Corollary 13

$$c^T(X)|_p = \sum b_i t^i$$

for gradations smaller than n . The top gradation of the equivariant Chern class is equal to t^n .

11 Top equivariant Chern class

It remains to prove Theorem 14, which is the key to the whole procedure of computing equivariant Chern classes. The top equivariant Chern class localized at a fixed point does not depend seriously on the set itself, but only on whether the point belongs to the set or not.

Theorem 14 *Suppose that X is a T -variety, not necessarily smooth. Let $p \in X$ be an isolated fixed point. Then the top gradation of the class $c^T(X)$ restricted to $\{p\}$ is equal to the product of weights appearing in the tangent representation at p .*

By additivity of Chern classes it follows that if $p \notin X$ then $(c_{top}^T(X))|_p = 0$. The core of the proof is the basic equation of Euler characteristics

$$\chi(X) = \chi(X^T).$$

Nevertheless the argument demands some formal manipulations.

Let N be a smooth complete variety with a torus action. Let us decompose $N = \sqcup_{\alpha \in A} N_\alpha$ into connected components. Let $i_\alpha : N_\alpha \rightarrow N$ be the inclusion. First of all we note that

Proposition 15

$$c_{top}^T(N) = \sum_{\alpha \in A} (i_\alpha)_* (c_{top}(N_\alpha)) \in H_T^{2 \dim(N)}(N). \quad (14)$$

Proof. The proof is the straight forward application of the formula (1) since

$$i_\alpha^*(c_{top}^T(N)) = e_\alpha \cdot c_{top}(N_\alpha) \in H_T^*(N_\alpha) = H_T^*(\{pt\}) \otimes H^*(N_\alpha).$$

□

Proof of Theorem 14. We will argue that for any equivariant constructible function $F : M \rightarrow \mathbb{Z}$ the equality holds

$$f_*(c_{top}^T(F))|_p = F(p) e_p \in H_T^{2 \dim(M)}(\{p\}).$$

It is enough to show that statement for M complete and the constructible function of the shape $F = f_*(\mathbb{1}_N)$ for an equivariant map $f : N \rightarrow M$ from a smooth complete variety N . (We can assume that by the usual argument which is available thanks to [32] and T -equivariant resolution of singularities [11].) It remains to prove that for an isolated fixed point $p \in M$

$$f_*(c_{top}^T(N))|_p = \chi(f^{-1}(p)) e_p \in H^{2 \dim(M)}(\{p\}). \quad (15)$$

Let i_α be as above and $f_\alpha = f i_\alpha : N_\alpha \rightarrow M$. By Proposition 15 we have

$$f_*(c_{top}^T(N)) = \sum_{\alpha \in A} f_*(i_\alpha)_*(c_{top}(N_\alpha)) = \sum_{\alpha \in A} (f_\alpha)_*(c_{top}(N_\alpha)) \in H_T^{2 \dim(M)}(M).$$

Let $B \subset A$ be the set of components of N^T which are mapped to p . Then

$$\begin{aligned} f_*(c_{top}^T(N))|_p &= \sum_{\beta \in B} ((f_\beta)_*(c_{top}(N_\beta)))|_p \\ &= \sum_{\beta \in B} \chi(N_\beta) [p]|_p \\ &= \sum_{\beta \in B} \chi(N_\beta) e_p. \end{aligned} \quad (16)$$

The equation (15) holds because $\chi(f^{-1}(p)) = \chi(f^{-1}(p)^T)$ and $f^{-1}(p)^T = \bigsqcup_{\beta \in B} N_\beta$. □

12 Further directions of work

Several goals have not been reached so far. The most obvious directions of further work would be

- deduce positivity results,
- study global equivariant Chern classes of Schubert varieties and open cells,
- develop a suitable calculus of symmetric rational functions to handle expressions appearing in the Berline-Vergne formula for Grassmannians.

We hope to realize this program in future.

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