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Moduli spaces of sheaves and principal G-bundles

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1. Introduction

This survey is devoted to recent developments in constructing moduli spaces of geometric objects such as vector bundles or principal G-bundles on a fixed variety (or on a fixed family of varieties).

The choice of topics was dictated by the author's taste and it ignores much of the recent progress. In particular, we avoid analytic techniques and present only an algebraic part of the theory, with special emphasis on positive characteristic. Also we do not say anything about geometry of obtained moduli spaces. One of the reasons is that in many cases they were just constructed and they were not studied yet.

We will adopt the framework of the Geometric Invariant Theory (GIT) which forces us to restrict to a specially nice class of objects that can be fit to form a nice projective scheme. The well known idea behind is that in general there are either far too many objects to fit into a projective scheme (we need a parameter space) or they degenerate so badly that the moduli space would not be separated (quotients do not exist in general). People willing to sacrifice projective varieties using algebraic spaces or stacks still run into difficulties. They either talk about the moduli space of simple sheaves (this still restricts the class of sheaves and produces huge non-separated algebraic spaces) or they talk about the moduli stack of sheaves (which even after restricting to stable sheaves is not a Deligne–Mumford stack). In this last case it is sometimes possible to get useful information, usually looking at stratifications of the moduli stack coming from GIT (this presents the stack as a direct limit of moduli stacks) and using the full machinery of semistable sheaves developed to study projective moduli spaces. For example, the moduli stack is not separated although it satisfies the other half of the valuative properness criterion. Similarly, the moduli stack of semistable sheaves satisfies this part of the properness criterion (this follows from the construction of the moduli space although it was known before as Langton's theorem). Only after restricting to stable sheaves we get an Artin stack and in this case GIT says that it represents the group quotient. Obviously in this last case we have a very nice quasi-projective moduli scheme which universally corepresents the moduli functor. The above explanation obviously does not mean that moduli stacks are useless in the context of vector

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or G-bundles as they have many important applications, especially when studying moduli of G-bundles on curves (see, e.g., [**78**] for a survey).

The first examples of interesting moduli spaces of sheaves were the Jacobian variety of a curve or a Picard scheme of a variety. Both of them were known classically and they can be thought of as moduli spaces of line bundles with fixed numerical data on a variety. The quasi-projective moduli space of stable sheaves over a smooth complex curve was constructed by D. Mumford. Afterwards this moduli space was compactified by C. Seshadri by adding S-equivalence classes of semistable sheaves. With W. Haboush's proof of Mumford's conjecture this construction was known to work for curves defined over an arbitrary algebraically closed field (although it should be noted that Seshadri was able to construct this moduli space even without Mumford's conjecture). The case of higher dimensional varieties defined over an algebraically closed field of characteristic zero was first treated by D. Gieseker and M. Maruyama. Their construction was later vastly improved by C. Simpson whose approach was much simpler and worked also for singular varieties. This should be compared to the the theory of compactified Jacobians, which at that time was still not well understood in the case of very singular curves. The case of moduli spaces of semistable sheaves over a variety defined over an arbitrary field (or a universally Japanese ring such as $\mathbb{Z}$) was settled recently in [**38**] and [**39**]. The theory behind constructing moduli spaces describing basic properties of semistable sheaves is contained in the first part of the paper.

This part contains a survey of results about semistable sheaves like Bogomolov's inequality, restriction theorems and bounds on the dimension of the cohomology groups of sheaves. This theory was fairly complete in the case of varieties defined over the complex numbers. However, the general case was completed only recently and there are still a few open problems (some of them even over $\mathbb{C}$). Many of the discussed problems do not at the first sight seem very relevant to construction of moduli spaces as they deal with basic properties of coherent sheaves such as their Chern classes or dimensions of the cohomology groups. However, this is often the most difficult (or maybe just the "remaining") part, the rest being a standard GIT technology worked out a long time ago.

The second part of the paper is devoted to moduli spaces of (semistable) principal G-bundles. This theory was started by A. Ramanathan, who in his PhD thesis at the Tata Institute constructed the moduli space of semistable principal G-bundles over a smooth curve defined over an algebraically closed field of characteristic zero. This theory remained unpublished for many years and in the meantime there appeared many papers dealing with similar moduli spaces. In the last few years this theory was extended to higher dimensional complex varieties by T. Gomez, I. Sols and A. Schmitt.

However, the positive characteristic was mysterious even in the curve case. For semisimple groups and curves defined over a field of a large characteristic, the construction was done by V. Balaji and A. J. Parameswaran in [**4**].

Very recently, the construction of compactified moduli spaces of semistable principal G-bundles on smooth varieties defined over an arbitrary algebraically closed field was done in [**24**]. We will sketch the basic idea behind this construction.

There are many other interesting and very useful moduli spaces that are not dealt with in this paper like moduli spaces of parabolic sheaves, parabolic G-bundles, framed sheaves, Higgs bundles etc. One of the reasons behind is that while these moduli spaces offer a very good insight into some geometric problems like the Verlinde formula, non-abelian Hodge theory etc, their (algebraic) construction is technically more involved but it

follows a similar strategy as the construction of the more basic moduli spaces of sheaves and principal G-bundles.

We also did not go into the study of properties of the constructed moduli spaces but their behaviour in many cases is completely open.

Notation:

Throughout the paper X denotes a smooth projective variety defined over an algebraically closed field k of arbitrary characteristic. We also fix an ample line bundle $\mathscr{O}_X(1)$ on X. For convenience we denote by H any divisor corresponding to this line bundle.

2. Behaviour of semistability of sheaves

As usual in classification problems there are two parts of the problem of classification of sheaves: first we need to find some discrete invariants and then find a variety parametrizing sheaves with fixed invariants. The role of the discrete invariant is, as usual in algebraic geometry, played by the Hilbert polynomial. However, even on $\mathbb{P}^1$ vector bundles $\{\mathscr{O}_{\mathbb{P}^1}(n) \oplus \mathscr{O}_{\mathbb{P}^1}(-n)\}_{n\in\mathbb{N}}$ have the same Hilbert polynomial and they degenerate one into another so in any reasonable moduli space the corresponding points would not be closed and in the closure they would contain a unique closed point corresponding to $\mathscr{O}_{\mathbb{P}^1} \oplus \mathscr{O}_{\mathbb{P}^1}$. To avoid so bad moduli spaces we have to restrict to Gieseker semistable sheaves.

Let E be a rank r torsion free sheaf on X. By $P(E)$ we denote the Hilbert polynomial of E. By definition $P(E)(n) = \chi(X, E(nH))$. The highest coefficient of $P(E)$ is measured by the rank of E. The next coefficient of $P(E)$ is measured by the *slope* of E, which is defined as $\mu(E) = c_1(E)H^{n-1}/r$. Let us recall that E is called *Gieseker H-semistable* if for every subsheaf $F \subset E$ we have

$$\frac{P(F)}{\mathrm{rk}F} \leq \frac{P(E)}{\mathrm{rk}E}.$$

E is called *slope H-semistable* if and only if for every subsheaf $F \subset E$ we have

$$\mu(F) \leq \mu(E).$$

Obviously, if X is a curve then Gieseker and slope semistability are equivalent.

One of the first problems encountered when constructing the moduli space of Gieseker H-semistable sheaves is boundedness, i.e., existence of a flat family of Gieseker semistable sheaves with fixed Hilbert polynomial P that would be parametrized by a scheme of finite type over k and that would contain all Gieseker semistable sheaves with Hilbert polynomial P. Then dividing S by an appropriate equivalence relation gluing points corresponding to the same sheaf we get the moduli space. Obviously, this part requires a substantial amount of work as quotients by algebraic equivalence relations produce only algebraic spaces. In our case this part of the problem can be best solved using Geometric Invariant Theory (see Section 5).

Since Gieseker semistable sheaves are slope semistable, boundedness of Gieseker semistable sheaves follows from the following more general theorem:

THEOREM 2.1. *Let P be a fixed polynomial. Then there exists a k-scheme S of finite type and a coherent $\mathscr{O}_{S\times X}$-module E such that every slope H-semistable sheaf with Hilbert polynomial P is isomorphic to one of the sheaves in the set $\{E_{\{s\}\times X} : s \in S\}$.*

The history of proof of this theorem is quite involved. The curve case was done by M. Atiyah in [**2**]. In fact, he showed that indecomposable vector bundles on a curve form a bounded family (this is no longer true in higher dimensions). This implies that stable

vector bundles form a bounded family and then semistable also form a bounded family as they can be constructed as extensions of bundles from a bounded family. If X is a surface and the rank is two then boundedness was showed independently by F. Takemoto and D. Mumford (see [**82**]). The general surface case was done by M. Maruyama in [**45**] and later by D. Gieseker [**22**] using a different method (computations using the Riemann-Roch theorem). In general, for varieties defined over a field k of characteristic 0, the theorem follows from the Grauert–Mülich theorem (see Theorem 2.2 and the remarks below it) as was observed by H. Spindler in [**79**]. The proof in this case can be also found in [**49**]. The theorem was also known for sheaves of rank ≤ 4 (see the papers of Maruyama [**48**], [**49**] and T. Abe [**1**]). These proofs used the Riemann–Roch theorem computations and Maruyama's restriction theorem saying that slope semistable sheaves of rank smaller than the dimension of the variety restrict to slope semistable sheaves on a general hyperplane section. In general, the theorem was proven by the author in [**38**] by proving a new restriction theorem (see Theorem 2.16) related to Bogomolov's inequality. The final statement of the boundedness result can be stated in much more general way: in mixed characteristic for sheaves with bounded slope of the maximal destabilizing subsheaf on fibers of a morphism of finite type over a noetherian base scheme (see [**38**, Theorem 4.4]; see also Theorem 3.8).

If X is a curve then the theorem is rather easy. By Serre's duality one can easily find m such that for every semistable sheaf E with Hilbert polynomial P, we have $h^1(E(m)) = h^0(E^*(-m)\otimes\omega_X) = 0$ (the last equality follows from the definition of semistability) and similarly $h^1(m_P\otimes E(m)) = 0$ for every point $P\in X$ and its maximal ideal m_P. Then $E(m)$ is globally generated so it is a quotient of $\mathscr{O}_X^{P(m)} \simeq H^0(E(m))\otimes\mathscr{O}_X$. Then the theorem follows from Grothendieck's theorem saying the set of quotients of a fixed sheaf, when the Hilbert polynomial of the quotient is fixed, is bounded. This follows obviously from the fact that the Quot-scheme $\mathrm{Quot}(\mathscr{O}_X^{P(m)}, P)$ is of finite type over k, but logically the theorem is necessary for the construction of the Quot-scheme.

For higher dimensional varieties Theorem 2.1 is deduced from the curve case by means of some restriction theorems. This is where it is convenient to pass to slope semistability, as it is easier to prove restriction theorems for slope semistability than for Gieseker semistability (however, see Conjecture 3.13).

More generally, it is known that slope semistability is much better behaved with respect to natural operations such as pull backs, tensor products, etc. There are several explanations of this phenomenon that we will describe later on. In the complex case this follows from the Kobayashi–Hitchin correspondence.

So most of this section is devoted to study slope semistability in various situations.

First let us recall that each torsion free sheaf E has the uniquely determined Harder–Narasimhan filtration $0 = E_0 \subset E_1 \subset \cdots \subset E_m = E$ in which all quotients E_i/E_{i-1} are slope semistable (and hence torsion free) sheaves and

$$\mu(E_1) > \mu(E_2/E_1) > \cdots > \mu(E/E_{m-1}).$$

We set $\mu_{\max}(E) = \mu(E_1)$ and $\mu_{\min}(E) = \mu(E/E_{m-1})$. The difference between $\mu_{\max}(E)$ and $\mu_{\min}(E)$ measures how far is the sheaf E from being semistable. This difference plays an important role in this section.

2.1. Slope semistability in characteristic zero. In this subsection we assume that X is a smooth n-dimensional projective variety defined over an algebraically closed field k of characteristic zero.

2.1.1. *Pull backs and push forwards of slope semistable sheaves.* It is well known that if $f : Y \to X$ is a finite map then E is slope H-semistable if and only if f^*E is slope f^*H-semistable. The proof is by passing to the Galois cover and using uniqueness of the Harder–Narasimhan filtration to descent the subsheaves of this filtration taking invariants with the Galois group. It is also true that E is slope H-polystable if and only if f^*E is slope f^*H-polystable. Obviously, if f^*E is slope f^*H-stable then E is also slope H-stable, but the converse is false.

For example, let $f : C_2 \to C_1$ be an étale Galois covering of curves. Then any irreducible unitary representation of the Galois group of the covering gives rise to a stable vector bundle on C_1 (associated to the principal G-bundle f). The pull back of this vector bundle is a trivial vector bundle on C_2. This example uses the well known theorem of M. Narasimhan and C. Seshadri (see [**65**]): stable vector bundles on curves correspond to irreducible unitary representations of the fundamental group. It is also easy to construct similar example in higher dimensions. For example, for any finite group G there exists a smooth surface S with a free G-action so that S/G is smooth. Take an irreducible representation of G producing a stable vector bundle on S/G. Then the pull back of this vector bundle is trivial.

Push forwards of slope semistable sheaves usually are no longer semistable. For example Schwarzenberger (see [**75**, Theorem 3]) showed that every rank 2 vector bundle on a surface X is a push forward of a line bundle from some degree 2 covering of X. In fact, if $f : Y \to X$ is an arbitrary degree 2 ramified covering then $f_*\mathscr{O}_Y$ is no longer semistable. On the other hand, one can easily see that if $f : Y \to X$ is étale then the push foward of a slope f^*H-semistable sheaf is slope H-semistable. Moreover, if $f : Y \to X$ is an unramified Galois covering with solvable Galois group then the push forward of a slope f^*H-polystable sheaf is slope H-polystable (see [**83**, Proposition 1.7]).

It is worth pointing out that for a fixed finite map $f : Y \to X$ one can uniformly bound $\mu_{\max}(f_*E) - \mu_{\min}(f_*E)$ for semistable E using only invariants of X, Y, H and f (see Example 3.7). Hence f_*E is not far from being semistable.

A. Beauville in [**6**] conjectures that if $f : Y \to X$ is a finite morphism of curves and $g(X) \geq 2$ then f_*E is stable for a generic vector bundle E on Y.

2.1.2. *Tensor products of slope semistable sheaves.* One of the basic properties of slope semistability is that (a torsion free part of) the tensor product of slope semistable sheaves, exterior or symmetric powers of slope semistable sheaves etc. are still semistable. We will treat this property of semistability more systematically in Section 4, using principal bundles and their extensions via representations.

In the curve case the above property follows from the characterization of semistable vector bundles using ampleness. Namely, a vector bundle E on a smooth curve is semistable if and only if the $\mathbb{Q}$-vector bundle $E \otimes \det E^{-1/r}$ is nef (this means that the corresponding $\mathbb{Q}$-line bundle on the projectivization of E is nef, i.e., it is a limit of ample $\mathbb{Q}$-divisors in the Néron–Severi group). Since the tensor product of ample vector bundles is ample (see Hartshorne's paper [**27**]), we get the same statement for semistable vector bundles (see also, e.g., [**56**, Section 3]).

In the curve case the property follows also from previously mentioned correspondence between unitary representations of the fundamental group and semistable vector bundles of degree zero (see [**65**, Theorem 2]).

The higher dimensional case follows immediately from the curve case by applying, e.g., Flenner's restriction theorem (see Theorem 2.3). The original Maruyama's proof of this result was more complicated and used the Grauert–Mülich theorem (see Theorem 2.2).

Another approach is via the Hitchin–Kobayashi correspondence, which works directly in the higher dimensional varieties but this seems rather difficult if we want to apply it to torsion free sheaves (see [**5**]).

A very general approach due to S. Ramanan and A. Ramanathan [**66**] will be discussed in Section 4.

Some of the above approaches give a more detailed information when we take a tensor product of slope stable sheaves. In this case the obtained product is slope polystable, i.e., it is a direct sum of slope stable sheaves with the same slope (see also Theorem 4.6).

2.1.3. *Restriction theorems and Bogomolov's inequality.* Let H be a very ample divisor on X.

THEOREM 2.2. (the Grauert–Mülich theorem) *Let E be a slope H-semistable sheaf on X. Let $0 = E_0 \subset E_1 \subset \cdots \subset E_m = E_D$ be the Harder–Narasimhan filtration of the restriction of E to a general hyperplane $D \in |aH|$, where a is a positive integer. Then for every $i = 1, \ldots, m$ we have*

$$\mu_i - \mu_{i+1} \leq a^2 H^n,$$

where $\mu_i = \mu(E_i/E_{i-1})$.

This theorem was first proven by Grauert and Mülich for rank 2 bundles on $\mathbb{P}^2$. Then it was generalized by W. Barth, O. Forster, A. Hirshowitz, M. Maruyama, M. Schneider and H. Spindler (see [**31**, p. 77] for precise references).

The first theorem saying that a restriction of a (semi)stable sheaf to a general hyperplane section is (semi)stable was proven by V. Mehta and A. Ramanathan in [**52**]. Namely, they proved that in arbitrary characteristic restriction of a slope (semi)stable sheaf to a general hyperplane section of large degree is also slope (semi)stable (see [**52**] and [**54**]). Unfortunately, this does not imply any boundedness theorem as the degree of a hypersurface can depend on a sheaf and not only on its numerical invariants. In characteristic zero, Flenner in [**19**] generalized the method of proof of the Grauert–Mülich theorem to obtain the following effective version of the Mehta–Ramanathan theorem:

THEOREM 2.3. (Flenner's theorem) *Let a be a positive integer such that*

$$\frac{\binom{a+n}{a} - 1}{a} > H^n \max\left\{\frac{r^2-1}{4}, 1\right\} + 1.$$

If E is slope H-semistable then for a general divisor $D \in |aH|$ the restriction E_D is also slope H_D-semistable.

For the idea of proof see the proof of Theorem 2.20, where similar techniques are used.

Let E be a rank r torsion free sheaf on a smooth variety X. Then we set

$$\Delta(E) = 2rc_2(E) - (r-1)c_1(E)^2.$$

In general, this is just a class in the Chow ring but if X is a surface, we can treat it as a number.

In late seventies of the twentieth century F. Bogomolov proved in [**9**] the following remarkable inequality on Chern classes of semistable sheaves:

THEOREM 2.4. (Bogomolov's inequality) *Let X be a smooth projective surface. Then for any slope H-semistable sheaf E we have $\Delta(E) \geq 0$.*

Together with Flenner's theorem (in this case it is also sufficient to use Mehta–-Ramanathan's theorem) this implies the following more general theorem:

COROLLARY 2.5. *For any slope H-semistable sheaf E on a smooth n-dimensional variety X we have $\Delta(E)H^{n-2} \geq 0$.*

In [**38**] the author showed that this is a special case of the following restriction theorem:

THEOREM 2.6. *Let E be a slope H-semistable sheaf on X. Let $0 = E_0 \subset E_1 \subset \cdots \subset E_m = E_D$ be the Harder–Narasimhan filtration of the restriction of E to a general hyperplane $D \in |H|$. Then we have*

$$\sum_{i<j} r_i r_j (\mu_i - \mu_j)^2 \leq H^n \cdot \Delta(E) H^{n-2},$$

where $\mu_i = \mu(E_i/E_{i-1})$ and $r_i = \mathrm{rk}(E_i/E_{i-1})$ for $i = 1, \ldots, m$.

Unfortunately, the proof of this theorem, even assuming Bogomolov's inequality, is quite long and we did not introduce enough notation to show it. A basic idea in the characteristic zero case is that one should try to compute $\Delta(E)$ on the incidence variety using quotients of the Harder–Narasimhan filtration of the restriction $E|_D$.

Bogomolov's inequality can be also used to prove the following strong restriction theorem:

THEOREM 2.7. *Let E be a slope H-stable torsion free sheaf and let a be an integer such that*

$$a > \frac{r-1}{r}\Delta(E)H^{n-2} + \frac{1}{r(r-1)H^n}.$$

Then for every normal divisor $D \in |aH|$ such that E_D is torsion free, the restriction E_D is slope H_D-stable.

A slightly weaker version of this inequality (with worse bound on a) was first proven by F. Bogomolov (see [**10**], [**11**]) and generalized by A. Moriwaki (see [**59**, Theorem 3.1]). The above version can be found in [**38**, Theorem 5.2].

Assume that E is slope semistable with $\Delta(E)H^{n-2} = 0$ (this condition is satisfied, e.g., if E is flat). Then the above theorem implies that a restriction of E to any normal hyperplane is slope semistable. As another example consider $T_{\mathbb{P}^2}$. Then the theorem says that the restriction of $T_{\mathbb{P}^2}$ to any smooth curve of degree > 2 is stable (this shows that the coefficients in this theorem cannot be further improved).

2.2. Strong slope semistability in positive characteristic. In this section we assume that the base field k has positive characteristic p. In this case there exists another very useful notion of strong semistability, which in many ways looks like the notion of stability in the characteristic zero case.

Let $F^m : X \to X$ denotes the composition of m absolute Frobenius morphisms of X.

We say that E is *slope strongly H-semistable* if for all $m \geq 0$ the pull back $(F^m)^*E$ is slope $(F^m)^*H$-semistable. It is sufficient to check this property only for all sufficiently large m.

EXAMPLE 2.8. In this example we show that there is no effective way of checking if a vector bundle is strongly semistable: it can happen that all the Frobenius pull backs up to any fixed m_0 are semistable and the next one is unstable. This can happen even if we want to check strong semistability of sheaves in a fixed family of sheaves (on a fixed variety) parametrized by a scheme of finite type. The example is based on [**21**].

In [**21**] D. Gieseker showed for every $g \geq 2$ an example of a sequence $\{E_m\}_{m\in\mathbb{N}}$ of rank 2 vector bundles on some genus g curve C such that $E = E_1$ is not semistable and

$E_m = F^*E_{m+1}$ for every $m \geq 1$. Then for large m the bundles E_m are semistable but not strongly semistable.

So to get our example it is sufficient to take a family of all semistable rank 2 vector bundles with trivial determinant (this family is bounded).

2.2.1. *Semistability versus strong semistability.* It is easy to see that every semistable vector bundle on a curve of genus $g \leq 1$ is strongly semistable. This is a special case of the following theorem of V. Mehta and A. Ramanathan:

THEOREM 2.9. *(*[**53**, Theorem 2.1]*) If $\mu_{\max}(\Omega_X) \leq 0$ then all slope semistable sheaves on X are slope strongly semistable. If $\mu_{\max}(\Omega_X) < 0$ then all slope stable sheaves on X are slope strongly stable.*

PROOF. We prove only the first part, the proof of the other one is analogous. The proof is by induction on the rank. Assume that the assertion holds for sheaves of rank $< r$ and let E be a rank r sheaf contradicting the required assertion. We can assume that E is slope semistable but F^*E is not slope semistable. Let $E^1, \ldots, E^k$ be the quotients in the Harder–Narasimhan filtration of F^*E. By assumption $E^1 \subset F^*E$ does not descend under the Frobenius morphism. Hence by Cartier's descent (see [**33**, Theorem 5.1]) we have a non-trivial $\mathscr{O}_X$-homomorphism $E^1 \to E/E^1 \otimes \Omega_X$. So for some $i > 1$ we have a non-trivial $\mathscr{O}_X$-homomorphism $E^1 \to E^i \otimes \Omega_X$, which induces $E^1 \otimes (E^i)^* \to \Omega_X$.

By assumption E^1 and E^i are strongly semistable, so by Theorem 4.9 $E^1 \otimes (E^i)^*$ is also strongly semistable (see 2.2.3). But $\mu(E^1 \otimes (E^i)^*) > 0 \geq \mu_{\max}(\Omega_X)$, so there are no non-trivial maps from $E^1 \otimes (E^i)^*$ to Ω_X. □

This shows many examples of strongly semistable sheaves. For example, if the cotangent sheaf of X is contained in a trivial sheaf then any semistable sheaf on X is strongly semistable. This class of varieties includes homogeneous, abelian or toric varieties. However, in the curve case the theorem produces examples only on rational or elliptic curves. In this case another interesting method of constructing strongly semistable bundles is, as in characteristic zero, considering representations of the étale fundamental group. This was introduced by H. Lange and U. Stuhler in [**36**] for curves defined over an algebraic closure of a finite field. In this case they proved that bundles coming from representations of the algebraic fundamental group of a curve satisfy $(F^m)^*E \simeq E$ for some positive integer m. Such bundles are strongly semistable (Frobenius pull backs of the destabilizing subbundle would give subbundles of E of arbitrarily large degree).

On the other hand, Gieseker's example (see Example 2.8) shows that in every genus $g \geq 2$ there exist semistable bundles which are not strongly semistable. There is quite a lot of recent work studying such bundles for low rank, low genus or low characteristic (see, e.g., [**42**] and the references within). However, essentially nothing is known about the locus of such bundles in the moduli space of vector bundles of rank $r > 2$ on curves of genus $g > 2$ defined over fields of characteristic $p > 2$.

2.2.2. *Pull backs of slope strongly semistable sheaves.* Again as in the characteristic zero case if $f : Y \to X$ is a finite map and E is slope strongly H-semistable then f^*E is slope strongly f^*H-semistable. If f is separable then the proof is the same as in characteristic zero and in general we factor the morphism into separable and purely inseparable morphisms and use the definition of strong semistability.

2.2.3. *Tensor products of slope strongly semistable sheaves.* As in the characteric zero case tensor product of slope strongly semistable sheaves, exterior or symmetric powers of slope strongly semistable sheaves etc. are still strongly semistable.

In the curve case the above property again follows from the characterization of strongly semistable vector bundles using ampleness. Namely, a vector bundle E on a smooth projective curve C is strongly semistable if and only if a $\mathbb{Q}$-vector bundle $E \otimes \det E^{-1/r}$ is nef. The proof of this fact is essentially the same as in the characteristic zero case (a nice account of this theory can be found in [**60**, Section 7]).

However, in general it is not known if there exists any restriction theorem for slope strong semistability (see however Theorem 2.20), so the above proof gives no information about higher dimensions. Fortunately, in general the proof of Ramanan and Ramanathan works (see Section 4 for more details).

2.2.4. *Restriction theorems and Bogomolov's inequality.* It is well known that the Grauert–Mülich theorem fails in positive characteristic (also for strongly semistable sheaves). As an example one can look at restrictions of $(F^k)^*T_{\mathbb{P}^2}$ to lines (see also Example 3.2). But the author does not know of any counterexample to Flenner's theorem. In fact, it is not even known if restriction of a rank 2 semistable sheaf on $\mathbb{P}^2$ to a general conic is semistable.

As in characteristic zero we have the following restriction theorem:

THEOREM 2.10. ([**38**, Theorem 0.1]) *Let E be a strongly slope H-semistable sheaf on X. Let $0 = E_0 \subset E_1 \subset \cdots \subset E_m = E_D$ be the Harder–Narasimhan filtration of the restriction of E to a general hyperplane $D \in |H|$. Then we have*

$$\sum_{i<j} r_i r_j (\mu_i - \mu_j)^2 \leq H^n \cdot \Delta(E) H^{n-2},$$

where $\mu_i = \mu(E_i/E_{i-1})$ and $r_i = \mathrm{rk}(E_i/E_{i-1})$. In particular, $\Delta(E)H^{n-2} \geq 0$.

The inequality $\Delta(E)H^{n-2} \geq 0$ for a strongly slope semistable sheaf E was conjectured by A. Moriwaki in [**57**] and proved by him when X is a surface (but using strong boundedness theorem!; see Corollary 3.9). Note that now Mehta–Ramanathan's restriction theorem does not imply the corresponding inequality in higher dimensions, as we lack a restriction theorem for strongly semistable sheaves.

It is not clear if strongly slope semistable sheaves restrict to strongly slope semistable sheaves on hypersurfaces of large degree. In fact, there are very few known examples showing what happens to strong semistability when restricting to a hypersurface. It is known that a strongly semistable sheaf need not be strongly semistable on every hypersurface of a large degree (so there is no analogue of Theorem 2.7 for strong semistability; see [**12**]). But it is not known if there exists a restriction theorem even for a very general hypersurface of a large degree (theorems of this type say nothing for varieties defined over countable algebraically closed fields). Maruyama's restriction theorem (see [**48**, Theorem 3.1]) implies that if E is a rank $r < \dim X$ strongly semistable sheaf on X then its restriction to a very general hyperplane is strongly semistable. Apart from that result, existence of a similar restriction theorem is known only for some special vector bundles on special varieties. For example, $T_{\mathbb{P}^2}$ restricted to any smooth conic is semistable and it is stable when restricted to any smooth cubic curve (e.g., by Theorem 2.19). By Theorem 2.9, for vector bundles on cubic or conic curves semistability is equivalent to strong semistability. Therefore by [**31**, Lemma 7.2.10] $(T_{\mathbb{P}^2})_C$ is strongly semistable for a very general curve C of degree ≥ 2.

We will come back to the general problem of restricting strongly semistable sheaves after introducing some notation and necessary tools in 2.3.5.

2.2.5. *Strong semistability is not an open property.* In this subsection we show two examples related to two different questions of openness one could pose.

In the first example we show that there exists a family of semistable sheaves on a fixed variety such that strongly semistable sheaves do not form an open subset in the parameter space.

If we have a family $\mathscr{E}$ parametrized by S then the subset of S corresponding to strongly semistable bundles is just a countable intersection of open subsets of $U_m \subset S$ corresponding to bundles for which the m-th Frobenius pull-back is semistable. So in general one does not expect strong semistability to be an open property. But finding an explicit example is not completely obvious.

EXAMPLE 2.11. Let C be a genus $g = 2$ curve as in Example 2.8. Then the moduli space M of rank 2 semistable vector bundles on C with trivial determinant is isomorphic to $\mathbb{P}^3$ (see [**64**, Section 7, Theorem 2] for $\operatorname{char} k \neq 2$ and [**42**, Proposition 5.1] for $\operatorname{char} k = 2$). Consider the rational map $V_m : M \dashrightarrow M$ defined by sending the class of E to the class of $(F^m)^*E$. One can check that this is indeed a well defined algebraic rational map. It is easy to see that if a rank 2 vector bundle E is semistable but non-stable (such bundles are called *strictly semistable*) then it is strongly semistable. The set of all strictly semistable bundles forms a divisor D in M and V_m is a well defined morphism on D. Let U_m be the open set of points of M at which V_m is a morphism. Since D is ample (as $M \simeq \mathbb{P}^3$) and contained in U_m, we see that the set U_m has a finite complement (see [**32**, Theorem 3.2] for a similar argument). Hence the set $U = \bigcap_m U_m \subset M$ corresponding to strongly semistable vector bundles has a countable complement (but possibly finite) $Y = M - U$.

On the other hand Y contains points corresponding to semistable (hence stable) bundles in the sequence $\{E_m\}$ described in Example 2.8. It is easy to see that no two bundles in this sequence are isomorphic since this would imply that all these bundles were strongly semistable. Hence the set Y is infinite.

This implies that if k is uncountable then in the family of all semistable rank 2 vector bundles on C with trivial determinant, the set of points parametrizing strongly semistable vector bundles is not open in the parameter space.

Note that if k is countable it is not clear that U is dense in M. This is known in $\operatorname{char} k = 2$ case (see [**43**, Proposition 8.1]).

The second example is related to the following question posed by Y. Miyaoka (see [**56**, Problem 5.4]). Suppose that $C \to \operatorname{Spec} \mathbb{Z}$ is a curve and E is a bundle on C for which $E_{\mathbb{Q}}$ is semistable. Is the set of points of $\operatorname{Spec} \mathbb{Z}$ such that E_k is strongly semistable still open? The following example shows that the answer to this question is negative. In fact, even the set of points where the first Frobenius pull back is semistable is not necessarily open.

EXAMPLE 2.12. Let $\pi : \mathbb{P}^2_{\mathbb{Z}} \to \mathbb{P}^2_{\mathbb{Z}}$ be a degree 4 covering given by $[x : y : z] \to [x^2 : y^2 : z^2]$. Let $C \subset \mathbb{P}^2_{\mathbb{Z}}$ be a curve given by $x^d + y^d + z^d = 0$. Set $E = (\pi^* T_{\mathbb{P}^2_{\mathbb{Z}}})_C$. By Theorem 2.7 we see that if $d \geq 7$ the bundle $E_{\mathbb{Q}}$ is semistable at the generic point $\bar{\mathbb{Q}}$ of $\mathbb{Z}$.

The following computation is due to H. Brenner (see [**13**]). Let us consider E_k, where k is a field of characteristic p. Constructing a non-trivial section of an appropriate twist of $(F^e)^*E_k$ it is not difficult to show that E_k is not strongly semistable if $p^e \equiv s \pmod{d}$ and $2s < d < 3s$ for some s coprime to d. One can use Dirichlet's theorem about existence of infinitely many primes in an arithmetic progression to show that this condition is satisfied for infinitely many primes p.

A stronger computation, using (the same as above) examples of C. Han and P. Monsky, can be found in the recent paper of V. Trivedi (see [**84**, the last paragraphs]). Let us set $E' = (T_{\mathbb{P}^2_{\mathbb{Z}}})_C$ and take $d = 4$. Then Trivedi shows that the bundle E'_k is semistable whereas $F^*E'_k$ is not semistable if $p \equiv \pm 3 \pmod 8$. But if $p \equiv \pm 1 \pmod 8$ then E'_k is strongly semistable.

2.3. Slope semistability in positive characteristic.

2.3.1. *Pull backs of slope semistable sheaves.* If $f : Y \to X$ is a finite separable morphism and E is slope H-semistable then f^*E is slope f^*H-semistable. Gieseker's example (see Example 2.8) shows that this is no longer true for non-separable morphisms (although see Theorem 2.9). The degree of instability of Frobenius pull backs of a sheaf is measured by the following numbers:

$$L_{\max}(E) = \lim_{k\to\infty} \frac{\mu_{\max}((F^k)^*E)}{p^k}$$

and

$$L_{\min}(E) = \lim_{k\to\infty} \frac{\mu_{\min}((F^k)^*E)}{p^k}.$$

Clearly, $L_{\min}(E) = -L_{\max}(E^*)$ so to simplify notation we will restrict to studying $L_{\max}$. By definition, E is strongly slope semistable if and only if $\mu(E) = L_{\max}(E)$.

Note that the sequence $\{\frac{\mu_{\max}((F^k)^*E)}{p^k}\}$ is non-decreasing so the limit exists although it could be infinity. The fact that the sequence is bounded can be seen in the following way: there exists a very ample divisor A such that $E^*(A)$ is globally generated. Then $(F^k)^*(E^*(A)) = ((F^k)^*E)^* \otimes \mathscr{O}_X(p^kA)$ is also globally generated. Therefore $(F^k)^*E$ embedds into a direct sum of copies of $\mathscr{O}_X(p^kA)$, so $\frac{\mu_{\max}((F^k)^*E^*)}{p^k} \le \mu(\mathscr{O}_X(A))$ and the limit is finite.

The following theorem implies that in fact the sequence $\{\frac{\mu_{\max}((F^k)^*E)}{p^k}\}$ stabilizes, so that $L_{\max}(E)$ and $L_{\min}(E)$ are well defined rational numbers:

THEOREM 2.13. (see [**38**, Theorem 2.7]) *For every torsion-free sheaf E there exists some $k \ge 0$ such that all quotients of the Harder–Narasimhan filtration of $(F^k)^*E$ are slope strongly semistable.*

One can also show that on a fixed variety X one can uniformly bound the degree of instability. Let us set

$$L_X = \begin{cases} \dfrac{L_{\max}(\Omega_X)}{p} & \text{if } \mu_{\max}(\Omega_X) > 0, \\ 0 & \text{if } \mu_{\max}(\Omega_X) \le 0. \end{cases}$$

PROPOSITION 2.14. (see [**38**, Corollary 6.2]) *For any rank r semistable sheaf E we have*

$$L_{\max}(E) - L_{\min}(E) \le (r-1)L_X.$$

In particular, for any sheaf E we have

$$L_{\max}(E) - \mu_{\max}(E) \le (r-1)L_X.$$

In case $\mu_{\max}(\Omega_X) \le 0$ the above proposition is just a reformulation of Theorem 2.9.

2.3.2. *Push forwards of slope semistable sheaves.* In [**51**] V. Mehta and C. Pauly showed that for a curve X of genus $g \geq 2$, if E is semistable then F_*E is also semistable. In the higher-dimensional case this problem was considered by X. Sun in [**85**], who in the curve case proved that stability of E implies stability of F_*E (if E is a line bundle this was first proven by H. Lange and C. Pauly in [**42**, Proposition 1.2]). In particular, Beauville's conjecture holds for purely inseparable morphisms of curves.

2.3.3. *Tensor products of slope semistable sheaves.* It is easy to see that the tensor products of slope semistable sheaves, exterior or symmetric powers of slope semistable sheaves etc. need not be slope semistable. For example if E is a degree zero semistable vector bundle such that F^*E is not semistable then S^pE is not semistable as $F^*E \subset S^pE$. Then $E^{\otimes p}$ is not semistable, as S^pE is its quotient.

But by 2.2.3 and Theorem 2.13 we have

$$L_{\max}(E_1 \otimes E_2) = L_{\max}(E_1) + L_{\max}(E_2)$$

for any torsion free sheaves E_1 and E_2. So Theorem 2.9 and Proposition 2.14 imply that we can bound the degree of instability of tensor products (and also of symmetric powers etc.) of semistable sheaves:

PROPOSITION 2.15. *Let E_1 and E_2 be slope semistable sheaves of rank r_1 and r_2, respectively. Then*

$$\mu(E_1) + \mu(E_2) \leq \mu_{\max}(E_1 \otimes E_2) \leq \mu(E_1) + \mu(E_2) + (r_1 + r_2 - 2)L_X.$$

2.3.4. *Restriction theorems and Bogomolov's inequality.* As in characteristic zero we have the following restriction theorem:

THEOREM 2.16. ([**38**, Theorems 3.1 and 3.2]) *Let E be a rank r torsion free sheaf on X. Let $0 = E_0 \subset E_1 \subset \cdots \subset E_m = E_D$ be the Harder–Narasimhan filtration of the restriction of E to a general hyperplane $D \in |H|$. Then we have*

$$\sum_{i<j} r_i r_j (\mu_i - \mu_j)^2 \leq H^n \cdot \Delta(E) H^{n-2} + 2r^2 (L_{\max}(E) - \mu(E))(\mu(E) - L_{\min}(E)),$$

where $\mu_i = \mu(E_i/E_{i-1})$ and $r_i = \mathrm{rk}(E_i/E_{i-1})$.

The above statement is slightly stronger as we do not assume semistability of E.

We have the following version of Bogomolov's inequality, which works in arbitrary characteristic:

THEOREM 2.17. (see [**41**, Theorem 2.2]) *Let H be an ample line bundle over X. If E is a rank r slope H-semistable sheaf then*

$$H^n \cdot \Delta(E) H^{n-2} + \frac{r^2(r-1)^2}{4} {L_X}^2 \geq 0.$$

For ranks $r = 2, 3$, if $\mu_{\max}(\Omega_X) > 0$ then one can replace in the above theorem L_X with $\frac{\mu_{\max}(\Omega_X)}{p}$ (see [**58**, Theorem 3] and [**63**, Proposition 1.4 and Theorem 1.7]). This follows from the fact that one of the factors of the Harder–Narasimhan filtration of a non-semistable sheaf of rank 2 or 3 has rank 1 and hence it is easy to compute $\mu_{\max}$ for tensor products with such a sheaf. In these cases, the theorem works also if one assumes that H is only nef.

In general, to get a good bound on $\Delta(E)H^{n-2}$ we need to use the Ramanan–Ramanathan theorem (Theorem 4.9). Hence the theorem a priori works only for ample polarizations. In case the polarization is nef, the statement would follow if we knew that $L_{\max}$ is continuous

under change of polarizations (but this is probably false). In general, we have the following slightly weaker theorem:

THEOREM 2.18. (see [**38**, Theorem 5.1 and Corollary 2.5]) *Let A be a nef line bundle over X such that $T_X(A)$ is globally generated. Let L be a nef line bundle. If E is a rank r slope L-semistable sheaf then*

$$L^n \cdot \Delta(E)L^{n-2} + \frac{r^2(r-1)^2}{(p-1)^2}(AL^{n-1})^2 \geq 0.$$

In particular, if E is strongly slope L-semistable then $\Delta(E)L^{n-2} \geq 0$.

As in characteristic zero Theorem 2.17 can be used to prove the following strong restriction theorem (this is a slight improvement of [**38**, Theorem 5.2]):

THEOREM 2.19. *Let E be a slope H-stable torsion free sheaf and let a be an integer such that*

$$a > \frac{r-1}{r}\Delta(E)H^{n-2} + \frac{1}{r(r-1)H^n} + \frac{r(r-1)^3}{4H^n}L_X^2.$$

Then for every normal divisor $D \in |aH|$ such that E_D is torsion free, the restriction E_D is slope H_D-stable.

2.3.5. *Restriction theorem for strong semistability.* Existence of a restriction theorem for strong semistability for a very general member of sufficiently ample linear system is a folklore conjecture among specialists in the field. Below we present the first such "general" restriction theorem ([**56**, Proposition 5.2] would imply existence of a very strong restriction theorem for ordinary abelian varieties but this proposition is stated without proof and it is probably false). The proof is similar to the proof of the Grauert–Mülich theorem (Theorem 2.2). Our approach is to use a non-separable descent, Theorem 2.19 and proof of Flenner's restriction theorem.

THEOREM 2.20. *Let (X,H) be a smooth n-dimensional ($n \geq 2$) polarized variety. Assume that $\Omega_X \hookrightarrow \mathcal{O}_X^N$ for some integer $N > 0$. Let E be an H-semistable torsion free sheaf of rank $r \geq 2$ on X. Let us take an integer a such that*

$$a > \frac{r-1}{r}\Delta(E)H^{n-2} + \frac{1}{r(r-1)H^n}$$

and

$$\frac{\binom{a+n}{a}-1}{a} > H^n \max\{\frac{r^2-1}{4}, 1\} + 1.$$

If $\mathrm{char}\, k > a$ *then the restriction E_D is strongly H-semistable for a very general divisor $D \in |aH|$.*

PROOF. By assumption $\mu_{\max}(\Omega_X) \leq 0$. Hence by Theorem 2.9 E is strongly H-semistable. By Theorem 2.19 we also know that E_D is semistable for a general divisor $D \in |aH|$. Assume that the restriction of E to a very general divisor in $|aH|$ is not strongly semistable. Let us take a minimal m such that the quotients of the Harder–Narasimhan filtration of the restriction of E to a very general divisor in $|aH|$ are strongly semistable.

Let Π denote the complete linear system $|aH|$. Let $Z = \{(D,x) \in \Pi \times X : x \in D\}$ be the incidence variety with projections $p : Z \to \Pi$ and $q : Z \to X$. Let Z_s denote the scheme theoretic fibre of p over the point $s \in \Pi$. Let $0 \subset E_0 \subset E_1 \subset \cdots \subset E_l = q^*((F^m)^*E)$ be the relative Harder–Narasimhan filtration of $(F^m)^*E$ with respect to p and set $E^i = E_i/E_{i-1}$. By definition this means that there exists a nonempty open subset U of Π such that all

factors $F_i = E_i/E_{i-1}$ are flat over U and such that for every $s \in U$ the fibres $(E_\bullet)_s$ form the Harder–Narasimhan filtration of $E_s = (q^*E)_{Z_s}$.

If the filtration $E_\bullet$ descends under the Frobenius morphism $Z \to Z$, then the descended filtration destabilizes $(F^{m-1})^*E$, which contradicts the choice of m. So the filtration $E_\bullet$ does not descend.

There exists a canonical connection ∇_{can} on $F^*((F^{m-1})^*E)$. By Cartier's theorem (see, e.g., [**33**, Theorem 5.1]) we know that there exist some i such that E_i is not preserved by $\nabla_{\text{can}} : (F^m)^*E \to \Omega_Z \otimes (F^m)^*E$. Then we get a non-zero $\mathscr{O}_X$-homomorphism

$$E_i \to \Omega_Z \otimes (F^m)^*E/E_i.$$

This implies that either

$$E_i \otimes ((F^m)^*E/E_i)^* \to \Omega_{Z/X}$$

is non-zero, or it is zero and then we have a non-trivial homomorphism

$$E_i \otimes ((F^m)^*E/E_i)^* \to q^*\Omega_X.$$

After restricting to a general fibre Z_s of p we see that

$$\mu_{\min}((E_i \otimes ((F^m)^*E/E_i)^*)_{Z_s}) \leq \max(\mu_{\max}((\Omega_{Z/X})_{Z_s}), \mu_{\max}((q^*\Omega_X)_{Z_s})).$$

By assumption, we have

$$\mu_{\max}((q^*\Omega_X)_{Z_s}) \leq 0.$$

Let us set $r_i = \mu((E^i)_{Z_s})$ and $\mu_i = \mu((E^i)_{Z_s})$. Then by the above, by assumptions about strong semistability and by 2.2.3 we have

$$\mu_i - \mu_{i+1} \leq \mu_{\max}((\Omega_{Z/X})_{Z_s}).$$

We know that $\mu_i > \mu_{i+1}$ and $\mu_i = \frac{\deg E^i}{r_i}$. We can write $\det E_i \simeq p^*\mathscr{L}_i \otimes q^*\mathscr{M}_i$ for some line bundles $\mathscr{L}_i$ on Π and $\mathscr{M}_i$ on X. Then $\deg E^i = \deg(\mathscr{M}_i|_{Z_s}) = aM_iH^{n-1}$, where $M_i = c_1\mathscr{M}_i$. Hence

$$\mu_i - \mu_{i+1} \geq \frac{aM_iH^{n-1}}{r_i} - \frac{aM_{i+1}H^{n-1}}{r_{i+1}} \geq \frac{a}{\max\{\frac{r^2-1}{4}, 1\}}.$$

Let K be the kernel of the evaluation map $H^0(\mathscr{O}_{\mathbb{P}^n}(a)) \otimes \mathscr{O}_{\mathbb{P}^n} \to \mathscr{O}_{\mathbb{P}^n}(a)$. Then from the proof of Flenner's theorem one gets

$$\mu_{\max}((\Omega_{Z/X})_{Z_s}) \leq H^n \cdot \mu_{\max}((K^*)_D),$$

where D is a general degree a hypersurface in $\mathbb{P}^n$. If the characteristic of the base field k is $> a$ then the rest of Flenner's proof also goes through. Namely, K is semistable and then by some simple computation one gets

$$\mu_{\max}((\Omega_{Z/X})_{Z_s}) \leq \frac{a^2H^n}{\binom{a+n}{a} - a - 1}.$$

This gives a contradiction with our assumptions on a. □

Note that the assumption on X is satisfied not only if the tangent bundle is globally generated (e.g., for abelian varieties or varieties of separated flags) but also in other cases, e.g., for smooth toric varieties.

The above proof raises the following question:

PROBLEM 2.21. *Is the kernel K of the evaluation map $H^0(\mathscr{O}_{\mathbb{P}^n}(a)) \otimes \mathscr{O}_{\mathbb{P}^n} \to \mathscr{O}_{\mathbb{P}^n}(a)$ semistable in arbitrary characteristic?* [1] *If not, find good estimates on $\mu_{\max}(K^*)$.*

A solution to this problem would allow to remove the annoying assumption on the characteristic of the base field from Theorem 2.20. However, note that even in characteristic zero it is not true that K_D is semistable for general $D \in |\mathscr{O}_{\mathbb{P}^n}(a)|$ (e.g., take $n = 2$ and $a = 1$).

2.4. Kodaira type vanishing theorems in positive characteristic. In this section X is a smooth projective variety defined over an algebraically closed field k of characteristic p. We assume that X has dimension $n \geq 2$ (otherwise all the theorems in this section become trivial). For a line bundle $\mathscr{L}$ we set $L = c_1\mathscr{L}$. Let us recall that a line bundle $\mathscr{L}$ is nef and big if and only if $LC \geq 0$ for every curve C and $L^n > 0$.

In [**70**] M. Raynaud showed that Kodaira's vanishing theorem fails in positive characteristic. More precisely, she showed a smooth projective surface X defined over an algebraically closed field of characteristic $p > 0$ and an ample line bundle $\mathscr{L}$ on X such that $H^1(X, \mathscr{L}^{-1}) \neq 0$. Since then there appeared many other examples violating Kodaira's vanishing theorem in positive characteristic. Nevertheless, as the following theorem shows such examples are rather special:

THEOREM 2.22. (Vanishing theorem) *Let $\mathscr{L}$ be a nef and big line bundle on X. If $H^0(X, \Omega_X \otimes \mathscr{L}^{-p^m}) = 0$ for every positive integer m then $H^1(X, \mathscr{L}^{-1}) = 0$.*

PROOF. We need the following

LEMMA 2.23. (see [**81**, 2.1, Critère, p. 178]) *Let $\mathscr{M}$ be a line bundle on X such that $H^0(X, \Omega_X \otimes \mathscr{M}^{-p}) = 0$ and $H^1(X, \mathscr{M}^{-p}) = 0$. Then $H^1(X, \mathscr{M}^{-1}) = 0$.*

PROOF. We have an exact sequence

$$0 \to \mathscr{O}_X \to F_*\mathscr{O}_X \to F_*\Omega_X.$$

Tensoring it with $\mathscr{M}^{-1}$ we get

$$0 \to \mathscr{M}^{-1} \to F_*(F^*\mathscr{M}^{-1}) \to F_*(\Omega_X \otimes F^*\mathscr{M}^{-1}).$$

By assumptions and the projection formula we have

$$H^0(X, F_*(\Omega_X \otimes F^*\mathscr{M}^{-1})) = H^0(X, \Omega_X \otimes \mathscr{M}^{-p}) = 0$$

and

$$H^1(X, F_*(F^*\mathscr{M}^{-1})) = H^1(X, \mathscr{M}^{-p}) = 0.$$

This easily implies $H^1(X, \mathscr{M}^{-1}) = 0$. □

Applying the lemma to $\mathscr{L}, \mathscr{L}^p, \mathscr{L}^{p^2}, \ldots$, we see that to prove our vanishing theorem it is sufficient to show that $H^1(X, \mathscr{L}^{-p^m}) = 0$ for large integers m. This immediately follows from the following weaker version of our theorem:

PROPOSITION 2.24. *Let $\mathscr{L}$ be a nef and big line bundle. Then $H^1(X, \mathscr{L}^{-m}) = 0$ for $m \gg 0$.*

[1] *Added in proof:* Recently, V. Trivedi in "Semistablity of syzygy bundles on projective spaces in positive characteristics", arXiv:0804.0547, gave a partial answer to this question.

PROOF. Assume that $H^1(X,\mathscr{L}^{-m}) \neq 0$. Since $Ext^1(\mathscr{L}^m,\mathscr{O}_X) = H^1(X,\mathscr{L}^{-m})$, there exists a non-split extension

$$0 \to \mathscr{O}_X \to E \to \mathscr{L}^m \to 0.$$

Using the well known Mumford's argument (which is the main part of Reider's method), we can check that E is slope L-semistable. Namely, assume that E is not L-semistable. Then there exists a destabilizing line bundle $\mathscr{M}$, which is a saturated subsheaf E (i.e., $E/\mathscr{M}$ is torsion free) and such that $ML^{n-1} > c_1EL^{n-1}/2 = mL^n/2 > 0$. Hence $Hom(\mathscr{M},\mathscr{O}_X) = 0$ and we have a non-trivial map $\mathscr{M} \to E/\mathscr{O}_X = \mathscr{L}^m$. Since the sequence is non-split this map cannot be an isomorphism and is thus given by some non-zero effective divisor $D \equiv mL - M$.

Let us choose an ample divisor H. Restricting to a general complete intersection of $(n-2)$ divisors from $|l(L+\varepsilon H)|$, we can easily see that for any positive rational number ε

$$(*) \qquad M(mL-M)(L+\varepsilon H)^{n-2} \leq c_2E(L+\varepsilon H)^{n-2} = 0.$$

Hence passing with ε to 0 we get

$$D(mL-D)L^{n-2} = M(mL-M)L^{n-2} \leq 0.$$

Therefore, using the Hodge index theorem and $(mL-2D)L^{n-1} = (2M-mL)L^{n-1} > 0$, we have

$$mDL^{n-1} \leq D^2L^{n-2} \leq \frac{(DL^{n-1})^2}{L^n} < \frac{m}{2}DL^{n-1},$$

which gives a contradiction with $DL^{n-1} \geq 0$.

Since E is L-semistable we can apply Theorem 2.18, which gives

$$L^n \cdot \Delta(E)L^{n-2} + 4\frac{(AL^{n-1})^2}{(p-1)^2} \geq 0.$$

Since $\Delta(E)L^{n-2} = -m^2L^n$, we get a contradiction for sufficiently large m. □

□

Theorem 2.22 was known when X is a smooth surface (see [**81**, Proposition 2.1]). If $\mathscr{L}$ is ample then the theorem follows from Serre's vanishing theorem and Lemma 2.23. Also some of corollaries of Theorem 2.22 were known earlier (see below).

Let us note that if k has characteristic zero, then Ω_X does not contain any line bundles of Kodaira dimension > 1 (this is a special case of Bogomolov's vanishing theorem). But if k has positive characteristic then it can happen that Ω_X contains even nef and big line bundles.

REMARK 2.25. Theorem 2.22 holds with essentially the same proof under weaker assumptions that L is nef and $L^2H_1\dots H_{n-2} > 0$ for some nef divisors $H_1,\dots,H_{n-2}$.

REMARK 2.26. One can also prove Theorem 2.22 assuming that X is normal, not smooth. The proof goes along the same lines but is more complicated. One of the main problems is inequality $(*)$: since $\mathscr{M}$ is a reflexive rank 1 sheaf, the inequality involves rational numbers (intersection product of two Weil divisors on a normal surface). However, since E is locally free one can try to pull back the sequence defining E to the resolution of singularities of a surface to get the inequality. This is in fact a special case of the theory of Chern classes of reflexive sheaves on normal surfaces developed by the author (see, e.g., [**37**, Proposition 2.11] for more details concerning the above inequality). Another problem is that Theorem 2.17 is valid only for smooth varieties. Here we could prove an appropriate version of Theorem 2.17 for sheaves on normal varieties using generic projections onto

projective spaces. A sufficient version of this theorem follows also from Theorem 3.6 (cf. the proof of Corollary 3.9).

As an immediate corollary from Theorem 2.22 we get a strengthening of Proposition 2.24:

COROLLARY 2.27. *Let $\mathscr{L}$ be a nef and big line bundle on X. Let m be a positive integer such that*

$$m > \frac{\mu_{\max}(\Omega_X)}{pL^n}.$$

Then $H^1(X,\mathscr{L}^{-m}) = 0$. In particular, if $\mu_{\max}(\Omega_X) < 2L^n$ then $H^1(X,\mathscr{L}^{-1}) = 0$.

In fact, the proof of Proposition 2.24, together with remarks after Theorem 2.17 produce another proof of this corollary.

The above corollary answers the question raised by J. Kollár in [**34**, II, Remark 6.2.4]. In particular, it implies that in [**34**, II, Theorem 6.2] one can replace "ample" with "nef and big". Let us state the following special case of this theorem:

COROLLARY 2.28. *Let L be a nef and big line bundle on X. Assume that X is covered by such a family of curves $\{C_t\}$ that*

$$((p-1)L - K_X)C_t > 0.$$

If X is not uniruled then $H^1(X,\mathscr{L}^{-1}) = 0$.

The above corollary is a generalization of [**58**, Corollary 5]. Moriwaki's vanishing theorem is slightly weaker as one can easily produce the family of curves taking general complete intersections of divisors in $|l_1(H_1 + \varepsilon H)|, \dots$ for some ample H. Moriwaki's assumptions on bigness of $\mathscr{L}$ are slightly weaker but our corollary can also be improved using Remark 2.25. So we could get a direct generalization of his result but we chose to show a simpler statement (with a simpler proof).

Applying [**34**, V, Lemma 5.1] we also get a vanishing theorem for uniruled varieties:

COROLLARY 2.29. *If X is separably uniruled then $H^1(X,\mathscr{L}^{-1}) = 0$ for any nef and big line bundle $\mathscr{L}$.*

One can reformulate the above corollary by saying that if there exists a nef and big line bundle $\mathscr{L}$ such that $H^1(X,\mathscr{L}^{-1}) \neq 0$ and if there exists a degree d uniruling of X then d is divisible by p.

COROLLARY 2.30. *If X is Frobenius split and non-uniruled then $H^1(X,\mathscr{L}^{-1}) = 0$ for any nef and big line bundle $\mathscr{L}$.*

The statement follows from Corollary 2.28 and equality $H^0(X,\omega_X^{1-p}) \neq 0$ which holds for any Frobenius split manifold.

For higher cohomology groups vanishing is more difficult to check. Let us just state the following proposition whose proof is left to the reader. The statement corrects [**81**, p. 183, Remarques iii)].

PROPOSITION 2.31. *Let $\mathscr{L}$ be an ample line bundle on X of dimension $n \geq 3$. Assume that $H^0(X,\Omega_X \otimes \mathscr{L}^{-p^m}) = 0$ for every $m \geq 0$, $H^0(X,\Omega_X^2 \otimes \mathscr{L}^{-p^m}) = 0$ for every $m \geq 1$ and $H^1(X,\Omega_X \otimes \mathscr{L}^{-p^m}) = 0$ for every $m \geq 1$. Then $H^2(X,\mathscr{L}^{-1}) = 0$.*

Hint: Use the technique of proof of Lemma 2.23 and the following exact sequence

$$0 \to \mathscr{O}_X \to F_*\mathscr{O}_X \to \ker(F_*\Omega_X \to F_*\Omega_X^2) \xrightarrow{C} \Omega_X \to 0,$$

where C is the Cartier operator.

3. General bounds on the cohomology groups and Chern classes of sheaves

3.1. Bounds on the number of sections. For each positive integer r we set

$$f(r) = -1 + \sum_{i=1}^{r} \frac{1}{i}.$$

An important role in the construction of moduli spaces of Gieseker semistable sheaves is played by the following theorem:

THEOREM 3.1. *Let E be a rank r torsion free sheaf on X. Then*

$$h^0(X,E) \leq \begin{cases} rH^n \binom{\frac{\mu_{\max}(E)}{H^n} + f(r) + n}{n} & \text{if } \mu_{\max}(E) \geq 0, \\ 0 & \text{if } \mu_{\max}(E) < 0. \end{cases}$$

It is easy to see that it is sufficient to prove the statement for semistable sheaves. In the characteristic zero case the method is to restrict to a hypersurface and count the number of sections using the Grauert–Mülich theorem:

EXAMPLE 3.2. Assume that $X = \mathbb{P}^2$ and E is a rank 2 semistable sheaf with $c_1E = 2k$. Then by the Grauert– Mülich theorem $E_L \simeq \mathscr{O}_{\mathbb{P}^1}(k) \oplus \mathscr{O}_{\mathbb{P}^1}(k)$ for a general line L. Then

$$\begin{aligned} h^0(E) \leq h^0(E(-1)) + h^0(E_L) \leq h^0(E(-2)) + h^0(E_L) + h^0(E_L(-1)) \leq \dots \\ \leq \sum_{i \geq 0} h^0(E_L(-i)) = \sum_{i \leq k+1} i = \binom{k+2}{2}. \end{aligned}$$

This method was known for a long time and used in this context (for example in Mumford's proof of the Castelnuovo–Mumford lemma in [**61**, Lecture 14]) but the nice statement (slightly weaker than the above) was formulated by C. Simpson and J. Le Potier (see [**77**, Corollary 1.7]).

In positive characteristic simple restriction is not sufficient for obtaining good results. For example if $E = S^2T_{\mathbb{P}^2} \otimes \mathscr{O}_{\mathbb{P}^2}(-3)$ then all the Frobenius pull backs $E_k = (F^k)^*E$ are rank 3 semistable sheaves with trivial determinant. But the restriction $(E_k)_L$ to any line is isomorphic to $\mathscr{O}_{\mathbb{P}^1}(-p^k) \oplus \mathscr{O}_{\mathbb{P}^1} \oplus \mathscr{O}_{\mathbb{P}^1}(p^k)$. Therefore $h^0((E_k)_L) = p^k + 2$, although the number of sections of E_k should be bounded independently of k (and p).

In general, in arbitrary characteristic Theorem 3.1 was proven by the author (see [**39**, Theorem 3.3]). The proof is by induction on dimension of the variety, the rank and the discriminant. The general idea is that if the restriction of a semistable sheaf to a hypersurface is not semistable then either the sheaf is an extension of semistable sheaves of similar slopes or one can use an elementary modification to get a sheaf with smaller discriminant Δ.

A similar method was used by the author to get some bounds on the Castelnuovo–Mumford regularity of sheaves on surfaces (see [**41**] and the next subsection).

3.2. Optimal bounds for cohomology of sheaves on surfaces. Let X be a smooth projective surface defined over an algebraically closed field. Let $H \in |\mathscr{O}_X(1)|$ be a very ample divisor. For a line bundle L on X we set

$$L' = \frac{L}{r} - \frac{K_X}{2}$$

and

$$\begin{aligned}\tau_L(r) =& \frac{r}{2H^2}\left((L'H)^2 - (L')^2 \cdot H^2\right) - \frac{L'H}{H^2} + \frac{r}{2}(H^2)^3 + \frac{r}{8}(K_X^2 + 3H^2)\\ &+ H^2 - r\chi(X, \mathscr{O}_X) - \frac{r+1}{2}.\end{aligned}$$

THEOREM 3.3. *Let E be a slope H-semistable torsion free sheaf of rank r on X.*
(1) If

$$l > \frac{\Delta(E)}{2r} + \tau_{c_1(E)}(r)$$

then $H^1(X, E(l)) = 0$.
(2) If E is locally free and

$$l > \frac{(rK_X - c_1E)H}{rH^2}$$

then $H^2(X, E(l)) = 0$.

The first part is [**41**, Theorem 0.1]. The second part follows from the Serre duality $h^2(X, E(lH)) = h^0(X, E^*(K_X - lH))$ and the remark that a semistable sheaf with negative slope has no sections. By Serre duality (1) in the above theorem implies explicit bounds for vanishing of $H^1(X, E(-l))$ for large l in case E is locally free. It is also easy to get bounds on $h^1(X, E)$ for locally free sheaves. One can simply use the Riemann–Roch formula and bounds on $h^0(X, E)$ and $h^2(X, E) = h^0(X, E^*(K_X))$.

It is easy to see that the above bounds are essentially optimal (for example in (1) of the above theorem one cannot correct the coefficient at $\Delta(E)$; see [**41**, Section 11]).

The above theorem was used in [**41**] to prove effective results on irreducibility of moduli spaces of sheaves on smooth surfaces (in arbitrary characteristic).

3.3. Bounds on the Castelnuovo–Mumford regularity in higher dimension. In this section we give a crude version of Theorem 3.3, which holds in arbitrary dimension.

Let X be an n-dimensional projective scheme over an algebraically closed field k and $H \in |\mathscr{O}_X(1)|$ an ample divisor on X. Let E be a coherent sheaf on X. Let d be the dimension of the support of E. Then there exist integers $a_0(E), \dots, a_d(E)$ such that

$$P(E)(m) = \chi(X, E(m)) = \sum_{i=0}^{d} a_i(E)\binom{m+d-i}{d-i}.$$

Let us assume that X is smooth. Then one can use the Riemann–Roch theorem to compute $a_i(E)$ using Chern classes of E and invariants of X. It is easy to see that $a_0(E) = rH^n$, where r is the rank of E. Assuming $r > 0$ we also get the following formulas:

$$\begin{aligned}a_1(E) &= \left(c_1(E) - \frac{r}{2}K_X\right)H^{n-1} - r\frac{(n+1)H^n}{2}\\ a_2(E) &= \frac{1}{2r}\left(c_1(E) - \frac{r}{2}K_X\right)^2 H^{n-2} - \frac{\Delta(E)H^{n-2}}{2r} - \frac{n}{2}\left(c_1(E) - \frac{r}{2}K_X\right)H^{n-1}\\ &\quad + r\frac{(n+1)(3n-2)H^n - 3K_X^2H^{n-2}}{24} + \frac{(K_X^2 + c_2(X))H^{n-2}}{12}.\end{aligned}$$

Let us recall that a coherent $\mathscr{O}_X$-module with $\operatorname{Supp} E = X$ *satisfies Serre's condition* (S_k) if for all $x \in X$

$$\operatorname{depth}(E_x) \geq \min(k, \dim \mathscr{O}_{X,x}).$$

Let us also recall that a sheaf E is called *m-regular* if $H^i(X, E(m-i)) = 0$ for $i > 0$. This definition is due to Mumford, who attributes it to Castelnuovo in his well known book on curves on surfaces.

Let $f : X \to S$ be a smooth projective morphism of relative dimension n of varieties of finite type over an algebraically closed field k. Let $\mathscr{O}_{X/S}(1)$ be an f-very ample line bundle on X.

Let $k \geq 2$ be an integer. Let $\mathscr{T}^k_{X/S}(r, c_1, \Delta; \mu_{\max})$ be the family of classes of such sheaves E on geometric fibres of f that

(1) E satisfies Serre's condition S_k,
(2) E is a rank r sheaf with $c_1(E)H^{n-1} = c_1$ and

$$\Delta(E)H^{n-2} - \left(c_1 E - \frac{r}{2}K_X\right)^2 H^{n-2} \leq \Delta,$$

(3) $\mu_{\max}(E) \leq \mu_{max}$.

Obviously, $\mathscr{T}^k_{X/S}(r, c_1, \Delta; \mu_{\max}) \subset \mathscr{T}^2_{X/S}(r, c_1, \Delta; \mu_{\max})$ (let us recall that on smooth varieties a sheaf is S_2 if and only if it is reflexive).

THEOREM 3.4. *Let $2 \leq k \leq n$ be an integer. There exist polynomials $P_{X/S}$, $Q_{X/S,i}$ and $R_{X/S,i}$ such that for any $E \in \mathscr{T}^k_{X/S}(r, c_1, \Delta; \mu_{\max})$ we have:*

(1) *$E(m)$ is m-regular for $m \geq P_{X/S}(r, c_1, \Delta, \mu_{\max})$,*
(2) *$H^{i-1}(X, E(-m)) = 0$ for $i \leq k$ and $m \geq Q_{X/S,i}(r, c_1, \Delta, \mu_{\max})$,*
(3) *$h^i(X, E(m)) \leq R_{X/S,i}(r, c_1, \Delta, \mu_{\max})$ for $1 \leq i \leq k-1$ and all m.*

Moreover, the set of Hilbert polynomials of sheaves in $\mathscr{T}^k_{X/S}(r, c_1, \Delta; \mu_{\max})$ is finite.

PROOF. In the proof we will restrict to semistable sheaves. Proof in the general case is the same but notation is more complicated. It uses Theorem 2.16 (see [**38**, Corollary 3.11' in Addendum]) instead of Theorem 2.19. Since by assumption $\mu_{\max} = c_1/r$, we omit it as a variable of polynomials.

To prove the theorem we will need the following generalization of the Enriques–Severi-Zariski lemma:

LEMMA 3.5. *Let E satisfy Serre's condition (S_k). Then $H^i(X, E(-l)) = 0$ for $i < k$ and $l \gg 0$.*

PROOF. If E satisfies (S_k) then $\mathscr{E}xt^q(E, \omega_X) = 0$ for $q > n-k$. Hence by Serre's duality $h^i(X, E(-l)) = \dim Ext^{n-i}(E, \omega_X(l)) = h^0(X, \mathscr{E}xt^q(E, \omega_X(l))) = 0$ for large l. □

The proof of the theorem is by induction on n. For $n = 2$ the theorem follows from the previous subsection. In fact, we can also give the same as proof of the inductive step below except that there is no c_2 on curves so notation is slightly different. This method gives very crude bounds on the polynomials.

Now assume that $n \geq 3$. Let E be a rank r coherent sheaf on some geometric fiber X_s. Assume that E is slope H-semistable and it satisfies (S_k). Let us set

$$L_{X/S} = \sup\{L_{X_s} : s \in S\}.$$

It is easy to see that $L_{X/S}$ is a well defined real number (use the fact that the relative tangent bundle tensored with an appropriate power of $\mathscr{O}_{X/S}(1)$ is relatively globally generated). Let

$$l \geq l_0 = \left\lceil \frac{r-1}{r}\Delta(E)H^{n-2} + \frac{r(r-1)^3}{4H^n}L_{X/S}^2 + 1 \right\rceil.$$

By Theorem 2.19 for a general divisor $D \in |O_{X_s}(l)|$ the restriction E_D is slope semistable. It also satisfies (S_k) (see, e.g., [**31**, Lemma 1.1.12 and Lemma 1.1.13]).

Set $\Pi = \mathbb{P}_S((f_*\mathscr{O}_X(l_0))^\vee)$. Let $Z \subset \Pi \times_S X$ be the relative incidence variety parametrizing over $s \in S$ pairs $\{(D,x) \in |O_{X_s}(l_0)| \times X_s : x \in D\}$. Let $T \subset \Pi$ be the subset parametrizing smooth divisors and let $p : Y \to T$ be the restriction of the projection $p : Z \to \Pi$. Then for a divisor $D \in T$ we have $c_1(E_D)H^{n-2} = l_0c_1(E)H^{n-1}$ and $\Delta(E_D)H^{n-3} = l_0\Delta(E)H^{n-2}$. By assumption we therefore have

(1) $E_D(m)$ is m-regular for $m \geq p = P_{Z/T}(r, l_0c_1, l_0\Delta)$.
(2) $H^{i-1}(D, E_D(-m)) = 0$ for $i \leq \max(k, n-1)$ and $m \geq q_i = Q_{Z/T,i}(r, l_0c_1, l_0\Delta)$,
(3) $h^i(X, E_D(m)) \leq r_i = R_{Z/T,i}(r, l_0c_1, l_0\Delta)$ for $1 \leq i \leq k-1$ and all m.

Using the short exact sequences

$$(*) \qquad 0 \to E(m-l_0) \to E(m) \to E_D(m) \to 0$$

we get

(1) $H^i(X, E(m+l_0)) = H^i(X, E(m))$ for $i \geq 2$ and all $m \geq p + (n-1)(l_0-1)$. By Serre's vanishing theorem this implies that for such m we have $H^i(X, E(m)) = 0$.
(2) $H^{i-1}(X, E(-m-l_0)) = H^{i-1}(X, E(-m))$ for $i \leq \max(k, n-1)$ and $m \geq q_i$. By Lemma 3.5 this implies that $H^{i-1}(X, E(-m)) = 0$ for $i \leq \max(k, n-1)$ and $m \geq q_i$.

For $i \leq \max(k, n-1) - 1$ we also have

$$h^i(E(m)) \leq h^i(E(m-l_0)) + h^i(E_D(m)),$$

so

$$h^i(E(m)) \leq \sum_{j \geq 0} h^i(E_D(m - jl_0)).$$

Since $h^i((E_D(m-jl_0)) = 0$ for $jl_0 \geq m + q_i$, we get

$$(**) \qquad h^i(E(m)) \leq \left[\left\lceil \frac{m+q_i}{l_0} \right\rceil\right]_+ r_i.$$

Let us take $m \geq p-1$. Then $h^1(E_D(m)) = 0$. Therefore $h^1(E(m)) \leq h^1(E(m-l_0))$ and if equality holds then $H^0(E(m)) \to H^0(E_D(m))$ is surjective.

Assume that $h^1(E(m+al_0)) = h^1(E(m+(a-1)l_0))$ for some $a \geq 0$ and consider the embedding $X_s \hookrightarrow \mathbb{P}^N$ given by $|\mathscr{O}_{X_s}(1)|$. Let $L \subset \mathbb{P}^N$ be the hyperplane defining D. For any integer $m \geq 1$ we have the commutative diagram

$$\begin{array}{ccc} H^0(E(m+al_0)) \otimes H^0(\mathbb{P}^N, \mathscr{O}_{\mathbb{P}^N}(bl_0)) & \xrightarrow{\alpha_1} & H^0(E(m+(a+b)l_0)) \\ \downarrow \pi_1 & & \downarrow \pi_2 \\ H^0(E_D(m+al_0)) \otimes H^0(L, \mathscr{O}_L(bl_0)) & \xrightarrow{\alpha_2} & H^0(E_D(m+(a+b)l_0)) \end{array}$$

in which the maps π_1 and α_2 are surjective (the last one by the Castelnuovo–Mumford theorem). It follows that π_2 is also surjective. Therefore $h^1(E(m+(a+b)l_0)) = h^1(E(m+(a-1)l_0))$ for all $b \geq 0$. By the Serre theorem this implies $h^1(E(m+(a-1)l_0)) = 0$.

Therefore the sequence $\{h^1(E(m+al_0))\}_{a\geq 0}$ is strictly decreasing until it reaches 0. In particular, $h^1(X,E(m+ll_0))=0$ for all $l>h^1(X,E(m))$. Together with $(**)$, this finishes the proof of existence of the polynomials $P_{X/S}$ and $R_{X/S,1}$.

The only remaining assertion is existence of $Q_{X/S,k}$ and $R_{X/S,k}$ in case $k=n$. But then E is locally free, so by the Serre duality $H^{n-1}(E(-m))=H^1(E^*(m)\otimes\omega_X)$ and we easily get existence of $Q_{X/S,n}$. Existence of $R_{X/S,n}$ is then easy as

$$h^{n-1}(E(-m))\leq h^{n-1}(E(-m-l_0))+h^{n-1}(E_D(-m))$$

and $h^{n-1}(E_D(-m))=h^0(E_D^*(m)\otimes\omega_D)$.

The fact that there are only finitely many Hilbert polynomials of sheaves in $\mathscr{T}^k_{X/S}(r,c_1,\Delta)$ can be seen as follows. All coefficients $a_0(E),\dots,a_{n-1}(E)$ are determined by E_D, so by assumption there is only finitely many possibilities for them. So it is sufficient to bound $a_n(E)$. The appropriate bound follows from

$$0\leq\chi(E(P(r,c_1,\Delta)))=h^0(E(P(r,c_1,\Delta)))\leq C,$$

where C depends only on r,c_1 and $P(r,c_1,\Delta)$ (see, e.g., Theorem 3.1). □

Proof of the above theorem is based on the proof of [**48**, Proposition 3.6]. A similar proof but using Flenner's restriction theorem was done in [**14**, Theorem A]. Note that the statement of F. Catanese and M. Schneider is weaker: in principle it contains a weaker assumption about singularities of X but see Theorem 3.6. The only difference is that in positive characteristic we cannot longer say that the polynomials depend only on the first two Chern classes of X as they also depend on L_X and the same invariant for complete intersections in X (obviously in characteristic zero we do not need this invariant).

In [**16**, Theorem 3.3], G. Elencwajg and O. Forster proved Theorem 3.4 for vector bundles on a projective space (again assuming $\mathrm{char}\,k=0$). This is already sufficient to recover Catanese–Schneider's statement using the projection method described in Example 3.7.

3.4. Bounds for cohomology of sheaves on singular varieties. Let $f:X\to S$ be as in the previous section: a smooth projective morphism of relative dimension n. Let $\mathscr{S}^k_{X/S}(d;a_0,a_1,a_2;\hat{\mu}_{\max})$, $k\geq 2$ be the family of sheaves on the geometric fibres of f such that E is a member of the family if

(1) the support of E is d-dimensional,
(2) E satisfies Serre's condition $S_{k,n-d}$ (see [**31**, p. 4]),
(3) $a_0(E)=a_0$, $a_1(E)=a_1$ and $a_2(E)\geq a_2$,
(4) $\hat{\mu}_{\max}(E)\leq\hat{\mu}_{max}$.

In general, if we have a singular projective variety X, then we can embedd it into a projective space $\mathbb{P}^N$ and then the family $\mathscr{S}^2_{\mathbb{P}^N}(d;a_0,a_1,\dots,a_n;\mu_{\max})$ contains the family of reflexive sheaves supported on X with bounded numerical invariants and bounded (generalized) slope of the maximal destabilizing subsheaf.

One can easily see that if $d=n$ then $a_0(E)=a_0$ means that we consider sheaves of fixed rank. The condition $a_1(E)=a_1$ means that we fix $c_1(E)H^{n-1}$ (use the formula for $a_1(E)$). Finally, using the explicit formula for $a_2(E)$ we see that the condition $a_2(E)\geq a_2$ implies that $\Delta(E)H^{n-2}-(c_1E-r/2K_X)^2H^{n-2}$ can be bounded from the above using only a polynomial expression in a_0,a_1,a_2 and numerical invariants of (X,H). In particular, the above families $\mathscr{S}^k_{X/S}(d;a_0,a_1,a_2;\hat{\mu}_{\max})$ generalize families $\mathscr{T}^k_{X/S}(r,c_1,\Delta;\mu_{\max})$.

Then we have the following version of Theorem 3.4.

THEOREM 3.6. *Let $2 \le k \le d$ be an integer. There exist (explicitly computable) polynomials $P'_{X/S}$, $Q'_{X/S,i}$ and $R'_{X/S,i}$ such that for any $E \in \mathscr{S}^k_{X/S}(d;a_0,a_1,a_2;\hat{\mu}_{\max})$ we have:*

(1) *$E(m)$ is m-regular for $m \ge P_{X/S}(a_0,a_1,a_2)$,*
(2) *$H^{i-1}(X,E(-m)) = 0$ for $i \le k$ and $m \ge Q_{X/S,i}(a_0,a_1,a_2)$,*
(3) *$h^i(X,E(m)) \le R_{X/S,i}(a_0,a_1,a_2)$ for $1 \le i \le k-1$ and all m.*

Moreover, the set of Hilbert polynomials of sheaves in $\mathscr{S}^k_{X/S}(d;a_0,a_1,a_2;\hat{\mu}_{\max})$ is finite.

This theorem can be derived from Theorem 3.4 by means of generic projections onto a smooth d-dimensional family (cf. proofs of [**31**, Theorem 3.3.1] and [**41**, Theorem 6.2]):

EXAMPLE 3.7. *Projection method.*

Suppose we want to prove Theorem 3.6. For simplicity assume that S is a point (and let us omit it in the notation). Take $E \in \mathscr{S}^k_X(d;a_0,a_1,a_2;\hat{\mu}_{\max})$.

Let us embedd $X \to \mathbb{P}^N$ by means of $|\mathscr{O}_X(1)|$ and take a generic linear projection onto $\mathbb{P}^d$. This gives a well defined finite morphism $\pi : Z \to \mathbb{P}^d$, where Z is the scheme-theoretical support of E. Then

$$\mu_{\max}(\pi_* E) - \mu_{\min}(\pi_* E) \le \hat{\mu}_{\max}(E) - \hat{\mu}_{\min}(E) + 2\,\mathrm{rk}\,\pi_*\mathscr{O}_Z$$

(see [**41**, Lemma 6.2.2]; cf. [**31**, Lemma 3.3.5]). Since $\mathrm{rk}\,\pi_*\mathscr{O}_Z \le (a_0(E))^2$, the theorem follows from equalities

$$H^i(X,E(k)) = H^i(\mathbb{P}^d, \pi_* E \otimes \mathscr{O}_{\mathbb{P}^d}(k)).$$

Let $\mathscr{S}_{X/S}(d;a_0,a_1,\ldots,a_d;\hat{\mu}_{\max})$ be the family of sheaves on the geometric fibres of f such that E is a member of the family if

(1) E is pure of dimension d (i.e., it satisfies Serre's condition $S_{1,n-d}$),
(2) $a_0(E) = a_0$, $a_1(E) = a_1$ and $a_i(E) \ge a_i$ for $i \ge 2$,
(3) $\hat{\mu}_{\max}(E) \le \hat{\mu}_{max}$.

This family generalizes the family of torsion free sheaves on a smooth variety, with bounded numerical invariants and bounded slope of the maximal destabilizing subsheaf. Similarly as above, we can prove an analogue of Theorem 3.6 but now (2) does not hold and the polynomials depend also on the remaining a_i's.

THEOREM 3.8. [**38**, Theorem 4.4] *The families $\mathscr{S}_{X/S}(d;a_0,a_1,\ldots,a_d;\hat{\mu}_{\max})$ and $\mathscr{S}^k_{X/S}(d;a_0,a_1,a_2;\hat{\mu}_{\max})$ are bounded.*

The second part follows from Theorem 3.6 and [**31**, Lemma 1.7.6]. The first one follows from the above mentioned analogue of Theorem 3.6. Note that in proof of the above theorem we did not use Kleiman's criterion for boundedness (see [**31**, Theorem 1.7.8]).

3.5. Bounds on Chern classes. First we show Moriwaki's argument proving Bogomolov's inequality (see [**57**, Theorem 1]):

COROLLARY 3.9. *Theorem 3.8 implies that $\Delta(E)H^{n-2} \ge 0$ for a strongly slope H-semistable sheaf E.*

PROOF. Passing to an appropriate covering of X we can assume that $c_1 E = 0$. If $\Delta(E)H^{n-2} < 0$ then

$$\Delta((F^k)^* E)H^{n-2} = p^{2k}\Delta(E)H^{n-2} < 0.$$

Hence the family $\{(F^k)^*E\}_{k\in\mathbb{N}}$ is a family of semistable sheaves, such that $c_1(G)H^{n-1}=0$ and $\Delta(G)H^{n-2}<0$ for any member of the family. By Theorem 3.8 such family is bounded so in particular $\Delta(G)H^{n-2}$ for G in this family is a finite set, a contradiction. □

The same argument shows that $c_i(E)H^{n-2}=0$ for all strongly semistable reflexive sheaves with $c_1(E)H^{n-1}=\Delta(E)H^{n-2}=0$. The analogue in characteristic zero can be found as [**76**, Theorem 2].

Theorem 3.8 immediately implies the following corollary:

COROLLARY 3.10. *Let X be a smooth manifold defined over an algebraically closed field k. Then there are only finitely many possibilities for Chern classes of slope semistable reflexive sheaves with fixed rank, first and second Chern classes.*

By Theorem 3.4 $E'=E(P_X(r,c_1,\Delta))$ is globally generated for suitable c_1 and Δ. By the result of W. Fulton and R. Lazarsfeld (see [**20**, Example 12.1.7]) for any partition λ of $k\le n$ we have

$$\Delta_\lambda(E')H^{n-k}\ge 0,$$

where Δ_λ is the Schur polynomial. These inequalities easily translate into corresponding inequalities for Chern classes of E. One can use them, e.g., to prove the following corollary, which in characteristic zero was proven by F. Catanese and M. Schneider (see [**14**, Theorem 3.11]; note that the proof requires inequalities from both sides). In case $X=\mathbb{P}^n$ and $\operatorname{char} k=0$, the corollary was proven by G. Elencwajg and O. Forster (see [**16**, Theorem 4.2]).

COROLLARY 3.11. *Let I be a multiindex and set $k=|I|$. There exist (computable) polynomials $P'_I(r,c_1,\Delta)$ and $P_I(r,c_1,\Delta)$ depending only on the numerical invariants of the manifold X such that for every semistable reflexive sheaf E of rank r with $c_1(E)H^{n-1}=c_1$ and $\Delta(E)H^{n-2}-(c_1E-r/2K_X)^2H^{n-2}\le\Delta$ on X we have*

$$P'_I(r,c_1,\Delta)\le c_I(E)H^{n-k}\le P_I(r,c_1,\Delta).$$

PROOF. The above mentioned inequalities imply that

$$0\le c_I(E')H^{n-k}\le c_1(E')^kH^{n-k}$$

(see [**15**, Corollary 2.6]). Since $c_I(E')$ can be expressed as a sum of $c_I(E)$ and some multiples of $c_{I'}(E)$ with $|I'|<k$, the corollary follows by induction on k. □

Note that we can also replace the condition $\Delta(E)H^{n-2}-(c_1E-r/2K_X)^2H^{n-2}\le\Delta$ with $(c_1E\cdot K_X/2-\operatorname{ch}_2E)H^{n-2}\le d_1$ and consider polynomials in variables r,c_1 and d_1.

J.-M. Drezet introduced the following logarithmic version of discriminant $\Delta(E)$. For a rank $r>0$ coherent sheaf E we write

$$\log\operatorname{ch}(E)=\log(r)+\sum_{i=1}^{n}(-1)^{i+1}\frac{1}{i!\,r^i}\Delta_i(E),$$

where $\Delta_i(E)\in A^i(X)\otimes\mathbb{Q}$ (Drezet's normalization is slightly different). For example, $\Delta_1(E)=c_1(E)$, $\Delta_2(E)=\Delta(E)$ and

$$\Delta_3(E)=3r^2c_3(E)-3r(r-2)c_1(E)c_2(E)+(r-1)(r-2)c_1^3(E).$$

Then as above there exist polynomials W_i, $i\ge 3$ depending only on X such that for any reflexive slope semistable sheaf we have

$$|\Delta_i(E)H^{n-i}|\le W_i(\Delta(E)H^{n-2}).$$

The polynomials do not depend on $c_1(E)$, since $\Delta_i(E)$ for $i\ge 2$ do not change when multiplying by a line bundle. It is interesting to check if one can choose the above polynomials

in a form independent of X (obviously this makes sense only in characteristic zero or for strongly semistable sheaves).

3.6. Gieseker semistability. Gieseker (semi)stability has much worse functorial properties than slope semistability. Even the dual of a Gieseker stable sheaf need not be stable (for example take on the projective space any Gieseker but not slope stable vector bundle of rank 3 with trivial determinant). It is also easy to see that there does not exist any restriction theorem for Gieseker stability. More precisely, it is not true that a restriction of a Gieseker stable sheaf to a (general) hypersurface of a very large degree is still Gieseker stable. As an example it is sufficient to take any Gieseker stable locally free sheaf on a surface, which is not slope stable. Then the restriction of this sheaf to any curve is not stable.

On the other hand, it is possible that a restriction of a Gieseker semistable sheaf to a general hypersurface of a sufficiently large degree is Gieseker semistable. In the surface case this fact immediately follows from restriction theorems. The above question was, to the author's knowledge, first posed by J. Le Potier (see remarks after [**43**, Théorème 2]). The positive answer to this question would allow to construct an analogue of the Donaldson–Uhlenbeck compactification ("the moduli space of slope semistable sheaves") for varieties of dimension greater than 2.

DEFINITION 3.12. Let E be a coherent sheaf on X with $\mathrm{Supp}E = X$. E is called *k-semistable* if and only if for every subsheaf $F \subset E$ we have

$$\frac{a_i(F)}{a_0(F)} \leq \frac{a_i(E)}{a_0(E)}$$

for $i \leq k$.

Similarly, one can also define k-stability (see [**31**, Section 1.6]; E is k-(semi)stable if and only if E is (semi)stable in $Coh_{n,n-k}(X)$, where $n = \dim X$). For $k = 1$ we get usual slope semistability. For $k = \dim X$ we get Gieseker semistability.

More generally than explained above, we expect that the following conjecture is true:

CONJECTURE 3.13. *(1) A restriction of a k-semistable sheaf to a general hypersurface of large degree is k-semistable. In particular, a restriction of a Gieseker semistable sheaf to a general hypersurface of large degree is Gieseker semistable.*

(2) A restriction of a k-stable sheaf to a general hypersurface of large degree is k-stable.

A strong version of the conjecture would be to find explicit bounds saying when the restriction is k-semistable. For $k = 1$ the strong version of the first part of the conjecture follows from Flenner's theorem in characteristic zero and Theorem 2.19 in general. The second part follows in this case from Theorems 2.7 and 2.19.

It is easy to see that a 2-semistable rank 2 vector bundle on $\mathbb{P}^n$, which is not 1-semistable, splits into a direct sum of line bundles. Hence in this case (1) of the conjecture holds. In fact, for rank 2 vector bundles on $\mathbb{P}^n$ Gieseker stability is equivalent to slope stability, so in this case also (2) holds. In general, (2) obviously implies (1).

4. Slope semistability of principal G-bundles

The moduli space of torsion free sheaves can be seen as a compactification of the moduli space of locally free sheaves. Rank r locally free sheaves can be treated as principal $\mathrm{GL}(r)$-bundles. More precisely, to every locally free sheaf one can associate the frame

bundle of the associated geometric vector bundle. On the other hand, to a principal $\mathrm{GL}(r)$-bundle one can associate the sheaf of sections of the geometric vector bundle associated to a principal $GL(r)$-bundle by a natural $\mathrm{GL}(r)$-action on k^r.

Now the problem is to construct the moduli space of (semistable) principal G-bundles for an arbitrary reductive group G and to find its compactification. Before considering this problem we study, as before, slope semistability of principal G-bundles.

In this section we fix a smooth polarized variety X, but for simplicity of the notation we usually avoid explicit using of the polarization.

The maximal open subset $U = U_{\mathscr{A}}$, where a torsion-free sheaf $\mathscr{A}$ is locally free, is big (i.e., its complement has codimension ≥ 2). In this case it makes sense to talk about semistability of $\mathscr{A}' = \mathscr{A}|_U$: this locally free sheaf is semistable if and only if $\mathscr{A}$ is slope semistable. One can easily see that this notion depends only on $\mathscr{A}'$ and it does not depend on the torsion free sheaf extending $\mathscr{A}'$. Similarly, we will consider principal G-bundles $\mathscr{P}$ defined on a big open subset of X and we will call them *rational G-bundles*. The situation is similar to rational maps: we usually not mention the set where $\mathscr{P}$ is defined to simplify notation. The maximal open subset to which $\mathscr{P}$ can be extended is denoted by $U_{\mathscr{P}}$.

One can also define a *rational vector bundle* as a vector bundle defined on a big open subset of X. In this case we can talk about the *degree* of a rational vector bundle. It is defined as the degree of the unique line bundle extending the determinant of a rational vector bundle.

A *rational reduction of structure group* of a rational G-bundle $\mathscr{P}$ to subgroup $H \subset G$ is a rational H-bundle $\mathscr{P}_H$ such that $\mathscr{P}$ is isomorphic (over some big open subset) to the extension $\mathscr{P}_H(G)$. Such rational reductions are in bijection with sections $\sigma : U \to (\mathscr{P}|_U)/H$, where U is a big open subset contained in $U_{\mathscr{P}}$.

DEFINITION 4.1. (A. Ramanathan, [**68**]) A rational G-bundle $\mathscr{P}$ is called *semistable* if for each rational reduction $\mathscr{P}_P$ to a parabolic subgroup $P \subset G$ we have $\deg \mathscr{P}_P(\mathfrak{p}) \leq 0$, where P acts on $\mathfrak{p}$ via the adjoint representation.

It can be checked that the above definition agrees with the corresponding definition for torsion free sheaves. As previously this definition in the curve case arose from some GIT semistability in appropriate parameter space.

PROPOSITION 4.2. *Let $\mathscr{P}$ be a rational G-bundle on X. Then the following conditions are equivalent:*

(1) *$\mathscr{P}$ is semistable.*
(2) *For any rational reduction $\mathscr{P}_P$ of structure group of $\mathscr{P}$ to a parabolic subgroup $P \subset G$ and any dominant character $\chi : P \to \mathbb{G}_m$ we have*
$$\deg(\mathscr{P}_P \times_\chi k) \leq 0.$$
(3) *For any section $\sigma : U \to (\mathscr{P}|_U)/P$ defined over a big open subset $U \subset U_{\mathscr{P}}$, we have*
$$\deg \sigma^* T_{(\mathscr{P}|_U/P)/U} \geq 0.$$

PROOF. Equivalence of (2) and (3) is proven in [**67**, Lemma 2.1]. To prove equivalence of (1) and (3) note that $\sigma^* T_{(\mathscr{P}|_U/P)/U} \simeq \mathscr{P}_P(\mathfrak{g}/\mathfrak{p})$, where P acts on $\mathfrak{g}/\mathfrak{p}$ via the adjoint representation. We also know that $\det \mathscr{P}(\mathfrak{g}) \simeq \mathscr{O}_X$ as the image of $Ad : G \to \mathrm{GL}(\mathfrak{g})$ lies in $\mathrm{SL}(\mathfrak{g})$. Hence the equivalence of (1) and (3) follows from the following short exact sequence of rational vector bundles:
$$0 \to \mathscr{P}_P(\mathfrak{p}) \to \mathscr{P}(\mathfrak{g}) \to \mathscr{P}_P(\mathfrak{g}/\mathfrak{p}) \to 0.$$

□

One of the first non-trivial problems is defining an analogue of the Harder–Narasimhan filtration. This was solved by K. Behrend in his PhD thesis:

THEOREM 4.3. (see [**7**, Theorem 7.3 and Proposition 8.2]) *Every rational G-bundle $\mathscr{P}$ has a rational reduction $\mathscr{P}_P$ to a parabolic subgroup $P \subset G$ such that*

(1) *For any rational reduction of $\mathscr{P}$ to a parabolic subgroup $Q \subset G$, we have* $\deg \mathscr{P}_Q(\mathfrak{q}) \leq \deg \mathscr{P}_P(\mathfrak{p})$.
(2) *P is not properly contained in any parabolic subgroup Q for which there exists a rational reduction $\mathscr{P}_Q$ to Q such that* $\deg \mathscr{P}_Q(\mathfrak{q}) = \deg \mathscr{P}_P(\mathfrak{p})$.

This reduction is uniquely determined (up to conjugacy of parabolic subgroups) and it is called the canonical reduction *(or the* Harder–Narasimhan reduction*). For such reduction, the extension of $\mathscr{P}_P$ to the Levi quotient $L = P/R_u(P)$ is a semistable rational L-bundle and for any dominant character $\chi : P \to \mathbb{G}_m$ we have*

$$\deg(\mathscr{P}_P \times_\chi k) \leq 0.$$

In characteristic zero existence of the Harder–Narasimhan reduction is rather easy and follows from the vector bundle case (which by now is also considered easy in itself). But in positive characteristic it is substantially more difficult so in this case the canonical reduction should be rather called *Behrend's reduction.*

4.1. Slope semistability in characteristic zero. In this case we can reduce studying rational G-bundles to the known case of rational vector bundles using the following proposition:

PROPOSITION 4.4. *A rational G-bundle $\mathscr{P}$ is semistable if and only if the rational vector bundle $\mathscr{P}(\mathfrak{g})$ (associated to $\mathscr{P}$ via the adjoint action of G on $\mathfrak{g}$) is semistable.*

The proposition follows from Theorem 4.6 (one implication) and existence of the Harder–Narasimhan reduction (the other implication).

This can be used to translate all results known for rational vector bundles to rational G-bundles. For example one can use it to directly define the Harder–Narasimhan reduction of rational G-bundles. Using this proposition one can easily generalize to rational G-bundles the facts about pull backs, the Grauert-Mülich and Flenner's restriction theorems. Alternatively, as in [**8**], one can rewrite known proofs of restriction theorems using the language of principal G-bundles.

The problems with generalizing Bogomolov's inequality or restriction theorems involving the second Chern class are slightly less obvious. One can uniquely extend a rational G-bundle to a principal G-bundle defined outside of a closed subset Y of codimension ≥ 3. Then for $i \leq 2$ the map $j : U = X - Y \to X$ induces isomorphisms of Chow groups $j^* : A^i(X) \to A^i(U)$. Hence it makes sense to talk about characteristic classes of rational G-bundles in $\bigoplus_{i \leq 2} A^i(X)$. In particular, it also makes sense to talk about classes $\Delta(\mathscr{P}(V)) \in A^2(X)$ for G-modules V. Since a (sheaf of sections of) rational vector bundle has a unique extension to a reflexive sheaf one can also define it as the discriminant of the unique reflexive extension of $\mathscr{P}(V)$.

Using this class one can generalize the theorems from sheaves to rational G-bundles. For example we have:

THEOREM 4.5. *Let $\mathscr{P}$ be a semistable rational G-bundle and let a be an integer such that*

$$a > \frac{r-1}{r}\Delta(\mathscr{P}(\mathfrak{g}))H^{n-2} + \frac{1}{r(r-1)H^n}.$$

Then for a general divisor $D \in |aH|$ the restriction $\mathscr{P}|_D$ is semistable.

Also the facts about tensor products and representations of semistable sheaves previously known for $\mathrm{GL}(r)$ can be generalized to rational G-bundles:

THEOREM 4.6. (the Ramanan–Ramanathan theorem, [**66**, Theorem 3.18]) *Let $\rho : G \to H$ be a homomorphism of reductive groups mapping the radical of G into the radical of H. Then for any semistable (polystable) rational G-bundle, the extended rational H-bundle $E(H)$ is also semistable (respectively, polystable).*

4.2. Strong semistability in positive characteristic. Now we assume that the base field k has positive characteristic p. As before we say that a rational G-bundle $\mathscr{P}$ is *strongly semistable* if for all $m \geq 0$ the pull back $(F^m)^*\mathscr{P}$ is semistable. Similarly as in characteristic zero we have:

PROPOSITION 4.7. (see [**40**, Corollary 2.8]) *A rational G-bundle $\mathscr{P}$ is strongly semistable if and only if the rational vector bundle $\mathscr{P}(\mathfrak{g})$ is strongly semistable.*

As above the proof follows from Theorem 4.9 and existence of the Harder-Narasimhan reduction. This proposition implies that the discriminant of the corresponding reflexive sheaf is non-negative (generalized Bogomolov's inequality). It also immediately implies the following analogue of Theorem 2.9:

THEOREM 4.8. *If $\mu_{\max}(\Omega_X) \leq 0$ then all semistable rational G-bundles on X are strongly semistable.*

Other proofs of this theorem can be found in [**55**, Theorem 4.1] and [**40**, Corollary 6.3].

Another application of the above proposition is that Theorem 2.20 works for strongly semistable rational G-bundles.

We also have the Ramanan–Ramanathan theorem for strong semistability:

THEOREM 4.9. (see [**66**, Theorem 3.23]) *Let $\rho : G \to H$ be a homomorphism of reductive groups mapping the radical of G into the radical of H. Then for any strongly semistable rational G-bundle, the extended rational H-bundle $E(H)$ is also strongly semistable.*

This theorem is particularly remarkable as in dimensions ≥ 2 there is no other proof of this theorem in the case of sheaves (i.e., corresponding to $G = \mathrm{GL}(r)$).

4.3. Semistability in positive characteristic. Existence of the Harder-Narasimhan reduction immediately implies that for a rational G-bundle $\mathscr{P}$, if $\mathscr{P}(\mathfrak{g})$ is semistable then $\mathscr{P}$ is semistable. But the converse is not true in general. To answer the question how far is $\mathscr{P}(\mathfrak{g})$ from being semistable for semistable $\mathscr{P}$ we need the following analogue of Theorem 2.13:

THEOREM 4.10. (see [**40**, Theorem 5.1]) *For every rational G-bundle $\mathscr{P}$ there exists some $m \geq 0$ such that the extension to the Levi quotient of the Harder–Narasimhan reduction of $(F^m)^*\mathscr{P}$ is strongly semistable. We call this reduction the* strong Harder–Narasimhan reduction.

Similarly as in the proof of Proposition 2.15 (but with quite complicated calculations) this can be used to prove the following:

THEOREM 4.11. ([**40**, Theorem 6.3]) *Let $\mathscr{P}$ be a semistable rational G-bundle. Assume that for some $l \geq 1$, $\tilde{\mathscr{P}} = (F^l)^* \mathscr{P}$ is not semistable and has strong Harder–Narasimhan reduction $\tilde{\mathscr{P}}_P$. Then*

$$0 < \mu(\tilde{\mathscr{P}}_P(V_\alpha)) \leq \mu_{\max}((F^{l-1})^* \Omega_X),$$

where V_α is an elementary $P-$module corresponding to some root α of P.

Note that the statement and the proof of [**40**, Theorem 6.3] contain errors and one should replace Q with P in a few places. This theorem allows to improve [**40**, Corollary 6.6] (which, as it uses Theorem 6.3, again contains easy to correct errors):

COROLLARY 4.12. *For any semistable rational G-bundle $\mathscr{P}$ on X we have*

$$0 \leq L_{\max}(\mathscr{P}(\mathfrak{g})) \leq L_X.$$

Using this one can easily prove the following theorem:

THEOREM 4.13. ([**40**, Theorem 8.4]) *Let $\rho : G \to H$ be a homomorphism of reductive groups mapping the radical of G into the radical of H. Then for any semistable rational G-bundle $\mathscr{P}$ we have*

$$0 \leq L_{\max}(\mathscr{P}_H(\mathfrak{h})) \leq C(\rho) L_X,$$

where $C(\rho)$ is an explicitly computable constant depeding only on ρ.

As a corollary one gets restriction theorems for rational G-bundles. For example, Theorem 2.16 (see [**38**, Corollary 3.11' in Addendum]) implies the following theorem:

COROLLARY 4.14. *Let $\mathscr{P}$ be a semistable rational G-bundle. Then for a general divisor $D \in |H|$ we have*

$$\mu_{\max}(\mathscr{P}|_D(\mathfrak{g})) - \mu_{\min}(\mathscr{P}|_D(\mathfrak{g})) \leq \sqrt{\frac{2H^n \cdot \Delta(\mathscr{P}(\mathfrak{g}))H^{n-2} + 4r^2(C(\rho))^2 L_X^2}{r}}.$$

Note that in general it is not known (in positive characteristic) if a restriction of a semistable rational G-bundle to a hypersurface of large degree is still semistable. But there is one easy case when one can get such a theorem. Before stating it we need to recall some information about low height representations (see [**50**] for a survey and references).

DEFINITION 4.15. Let $\rho : G \to \mathrm{SL}(V)$ be a faithful rational representation in characteristic p. A *height* $ht\,\lambda$ of a weight λ is the sum of coefficients when writing λ as a sum of simple roots. The representation ρ is called *low height* if $p > 2\max_i(ht\,\lambda_i)$, where λ_i are highest weights of simple G modules occuring in the socle filtration of V.

THEOREM 4.16. *Let $\mathscr{P}$ be a rational G bundle and let $\rho : G \to \mathrm{SL}(V)$ be a low height representation. Then $\mathscr{P}$ is semistable if and only if $\mathscr{P}(V)$ is semistable.*

As an immediate application of Theorem 2.19 and the above theorem we get the following:

THEOREM 4.17. *Let $\mathscr{P}$ be a semistable rational G-bundle and let $\rho : G \to \mathrm{SL}(V)$ be a low height representation. Let a be an integer such that*

$$a > \frac{r-1}{r}\Delta(\mathscr{P}(V))H^{n-2} + \frac{1}{r(r-1)H^n} + \frac{r(r-1)^3}{4H^n}L_X^2.$$

Then for a general divisor $D \in |aH|$, the restriction $\mathscr{P}|_D$ is semistable.

5. Moduli spaces

In this section we give sketches of construction of moduli spaces of sheaves and G-bundles.

5.1. Construction of moduli spaces of sheaves.

THEOREM 5.1. *There exists a coarse moduli scheme for Gieseker semistable sheaves with fixed Hilbert polynomial P.*

Sketch of the construction (after Simpson [**77**]*).* Let m be a large integer and let Q be the closure of the subset of the relative Quot-scheme parametrizing quotients of $\mathscr{H} = \mathscr{O}_X(-m)^{P(m)}$ with Hilbert polynomial P on geometric fibres of $f : X \to S$, parametrising pure sheaves of dimension $d = \deg P$.

Let $\mathscr{R}$ be the open subset of Q of those quotients $\mathscr{H}_s \to E$ for which E is Gieseker semistable on the fibre X_s, $s \in S$, and the induced map

$$H^0(X_s, \mathscr{H}_s(m)) \to H^0(X_s, E(m))$$

is an isomorphism. By Theorem 2.1 the family of Gieseker semistable sheaves is bounded, so if m is sufficiently large then the family of sheaves appearing quotients in $\mathscr{R}$ contains all Gieseker semistable sheaves. Then one can prove that $\mathscr{R}$ is equal to the set of points of Q which are semistable for the action of $\mathrm{GL}(P(m))$ with respect to the embedding determined by m. This follows by a long computation using the Hilbert–Mumford criterion and Theorem 3.1.

Now to construct the moduli space of semistable sheaves it is sufficient to find the quotient of $\mathscr{R}$ by $\mathrm{GL}(P(m))$. Under our assumptions this quotient exists by Mumford's results on GIT quotients and Haboush's proof of Mumford's conjecture. If the variety is defined over a universally Japanese base ring R then the result follows from Seshadri's extension of GIT: under this assumption the quotient is a projective scheme of finite type over R.

5.2. Moduli spaces of principal G-bundles over curves. This subject is particularly reach and we will just briefly describe its small part. The first algebraic construction of moduli spaces of semistable principal G-bundles over curves defined over a field of characteristic zero was done by A. Ramanathan. He used the adjoint representation $Ad : G \to \mathrm{GL}(\mathfrak{g})$ to reduce the study of G-bundles to vector bundles with fiber $\mathfrak{g}$. The basic idea was to reduce structure group of the $\mathrm{GL}(\mathfrak{g})$-bundle associated to the vector bundle to the adjoint form of G using automorphisms of the Lie algebra $\mathfrak{g}$ and then passing through the isogeny $G \to AdG$.

Unfortunately, his construction (see [**68**], [**69**]) was not published till 1996 after studying moduli spaces of G-bundles became important in connection with conformal field theory (publication was delayed because the author planned to revise the papers and not because of "small interest of mathematical community" as editors sometimes reject the papers!). At that time it was realized that in the curve case there is another very interesting approach using an old uniformization idea of Weil. This approach allowed to study the moduli stack of principal G-bundles as a quotient of an affine grassmannian (or a double quotient of a loop group). This led to much progress like proofs of the Verlinde formula etc. Obviously, this progress did not discard moduli spaces treated as algebraic varieties. They appeared to be an important tool especially when considered in the relative case. Their study for elliptic fibrations in the work of R. Friedman, J. Morgan, E. Witten and

A. Clingher allowed to explain physicists' duality between F-theory and heterotic string theory.

Going back to algebraic geometry, one should mention the work of G. Faltings [**17**], who gave a different to Ramanathan's algebraic construction of the moduli space of G-bundles on curves (again in characteristic zero). In Faltings' approach projectivity of moduli space does not follow from existence of the GIT quotient but it is proven separately as a semistable reduction theorem analogous to Langton's result in the sheaf case.

For semisimple groups the construction of quasi-projective moduli space of G-bundles for representations of low height (i.e., in a large characteristic; see Definition 4.15) was done by V. Balaji and A. J. Parameswaran in [**4**]. By Theorem 4.16 in this case vector bundles associated via such representation to semistable principal G-bundles are semistable. According to the introduction to Faltings' paper [**17**], this was the main reason why he also restricted to characteristic zero (although to the author and probably also to the authors of [**4**] this fact does not seem to be completely straightforward). In positive characteristic assuming this fact ideologically seems to be wrong as it often fails and omits the main problem instead of solving it. This approach was used before, e.g., in constructing the Harder–Narasimhan filtration for principal G-bundles or in proving Behrend's conjecture (see [**50**] for a survey), but as the author wants to stress this method does not seem to give any indication how to find a general solution to any of the above problems.

Another problem that appears is projectivity of the obtained moduli space. To obtain it V. Balaji and A. J. Parameswaran had to assume that the characteristic is even higher. This is related to the semistable reduction theorem, similar to Langton's theorem in the sheaf case. In some cases like moduli spaces of symplectic bundles or special orthogonal bundles (in characteristic different to 2 and again over curves) this was known for a long time (see, [**18**, Proposition 4.2]). A more general approach via affine grassmannians, working for almost all groups, is contained in a recent work of J. Heinloth (see [**28**]).

Recently, T. Gómez, A. Schmitt, I. Sols and the author (see [**23**] and [**24**]) constructed a quasi-projective moduli space of principal G-bundles in arbitrary characteristic. This moduli space is proven to be projective in large characteristic. Since the methods work in an arbitrary dimension, the construction will be sketched when talking about higher dimensions.

5.3. Generalized principal G-bundles. To compactify the moduli space of vector bundles in dimension ≥ 2 one needs to replace vector bundles by locally free sheaves and use torsion-free sheaves. Similarly, in order to compactify the moduli space of principal G-bundles we need to introduce new objects.

To define generalization of a principal G-bundle for simplicity we assume that G is a semisimple algebraic group. We will fix a faithful representation $\rho : G \to \mathrm{GL}(V)$ (this means that the kernel group scheme is trivial).

DEFINITION 5.2. A *principal ρ-sheaf* is a triple $(\mathscr{P}, \mathscr{A}, \psi)$ consisting of a rational G-bundle $\mathscr{P}$, a torsion free sheaf $\mathscr{A}$ of rank $\dim V$ such that $U_{\mathscr{A}} \subset U_{\mathscr{P}}$ and an isomorphism of locally free sheaves $\psi : \mathscr{P}(V^\vee)|_{U_{\mathscr{A}}} \to^{\simeq} \mathscr{A}|_{U_{\mathscr{A}}}$.

It is easy to see that $\mathscr{P}$ can be extended to a principal G-bundle exactly on $U_{\mathscr{A}^{**}}$. This is a (somewhat modified) generalization of T. Gómez and I. Sols definition [**26**, Definition 0.1] introduced in [**24**]. A modification is quite interesting: it means that on a smooth surface $\mathscr{P}$ can be defined on the whole surface and not only on a big open subset. This is particularly useful when one deals with non-faithul representations.

Another generalization of principal G-bundles, more technical but better suited to the construction of moduli spaces, was proposed by A. Schmitt in [**72**]:

DEFINITION 5.3. A *pseudo G-bundle* $(\mathscr{A},\tau)$ is a pair consisting of a torsion free sheaf $\mathscr{A}$ of rank $\dim V$ with $\det\mathscr{A}=\mathscr{O}_X$ and a non-trivial homomorphism $\tau : S^*(\mathscr{A}\otimes V)^G \to \mathscr{O}_X$ of $\mathscr{O}_X$-algebras.

By definition τ corresponds to a section $\sigma : X \to \mathbb{H}\mathrm{om}(\mathscr{A},V^\vee\otimes\mathscr{O}_X)//G$. If $\sigma(U_{\mathscr{A}})\subset \mathbb{I}\mathrm{som}(\mathscr{A},V^\vee\otimes\mathscr{O}_X)/G$ then we say that a pseudo G-bundle $(\mathscr{A},\tau)$ is a *singular G-bundle* (the names we use above differ from those used in [**72**] and [**73**] and follow [**24**]).

In case $(\mathscr{A},\tau)$ is a singular G-bundle we have the following base change diagram

$$\begin{array}{ccc} \mathscr{P} & \longrightarrow & \mathbb{I}\mathrm{som}(\mathscr{A}|_{U_{\mathscr{A}}},V^\vee\otimes\mathscr{O}_{U_{\mathscr{A}}}) \\ \downarrow & & \downarrow \\ U_{\mathscr{A}} & \longrightarrow & \mathbb{I}\mathrm{som}(\mathscr{A}|_{U_{\mathscr{A}}},V^\vee\otimes\mathscr{O}_{U_{\mathscr{A}}})/G \end{array}$$

in which $\mathscr{P}$ is a principal G-bundle defined over the big open subset $U_{\mathscr{A}}$. We also have a natural isomorphism $\mathscr{P}|_{U_{\mathscr{A}}}(\mathrm{GL}(V))\simeq \mathbb{I}\mathrm{som}(\mathscr{A}|_{U_{\mathscr{A}}},V^\vee\otimes\mathscr{O}_{U_{\mathscr{A}}})$ of principal $\mathrm{GL}(V)$-bundles. In this way we associate to $(\mathscr{A},\tau)$ a principal ρ-sheaf. Conversely, to any principal G-bundle over U and a torsion free sheaf $\mathscr{A}$ extending $\mathscr{P}(V^\vee)$ one can associate a unique σ as above. So there is a natural bijection between isomorphism classes of principal ρ-sheaves and singular G-bundles.

5.4. Moduli spaces of generalized G-bundles. In this subsection we construct a compactification of the moduli space for semistable principal G-bundles on a smooth projective variety X over an arbitrary algebraically closed field k. For a more recent survey containing many more results see the recent paper of A. Schmitt [**74**].

The moduli space of semistable principal G-bundles in case $k=\mathbb{C}$ was first constructed by D. Hyeon in [**30**] by a direct generalization of Ramanathan's results from the curve case. In this case the definition of semistability is particularly easy as it is equivalent with semistability of the associated bundle (this is no longer true for Gieseker semistability of pricipal ρ-sheaves). Later T. Gómez and I. Sols in [**26**] constructed the compactification of this moduli space using "principal G-sheaves", which are essentially a special case of the above defined principal ρ-sheaves when ρ is the adjoint representation. Independently, A. Schmitt in [**72**] constructed another compactification using pseudo G-bundles but he used arbitrary faithful representations of a semisimple group. Later in [**73**] he was able to check that the compactification contains only singular principal G-bundles. All these results required the characteristic zero assumption. Here we sketch the construction in general (see [**23**] and [**24**]). The construction is largely a generalization of the above constructions but there are many new obstacles.

It is convenient to use rational G-bundles for defining slope semistability for generalized G-bundles. Namely, we say that a principal ρ-sheaf (or a singular G-bundle) is *slope semistable* if the corresponding rational G-bundle is semistable. A corresponding notion of Gieseker semistability is much more difficult and the reader is refered to the original papers.

The following theorem holds for smooth varieties X defined over any algebraically closed field:

THEOREM 5.4. *(see* [**24**]*) There exists a projective moduli space parametrising semistable pseudo G-bundles $(\mathscr{A},\tau)$ with fixed Hilbert polynomial $P(\mathscr{A})$. It contains as an*

open set the moduli space of slope stable principal G-bundles over X. In case the characteristic is large the subset corresponding to singular principal G-bundles is closed in the moduli space of pseudo G-bundles.

Rough sketch of the proof. In the case of pseudo G-bundles the first problem is to find the parameter space. Note that $S^*(\mathscr{A}\otimes V)^G$ is finitely generated. So with a proper care we can replace a homomorphism of algebras $\tau : S^*(\mathscr{A}\otimes V)^G \to \mathscr{O}_X$ by another homomorphism of algebras $\tau' : S^*\mathscr{W} \to \mathscr{O}_X$, where $\mathscr{W}$ is associated to the frame bundle of $\mathscr{A}$ via some finite-dimensional representation W of $\mathrm{GL}(r)$. The latter is uniquely determined by a non-trivial homomorphism of sheaves $\mathscr{W} \to \mathscr{O}_X$.

In large characteristic for some non-negative a,b,c, the representation W is a quotient of $((k^r)^{\otimes a})^b \otimes \det((k^r)^*)^{\otimes c}$. Then one needs to study the associated decoration of type (a,b,c)

$$(\mathscr{A}^{\otimes a})^b \to (\det \mathscr{A})^{\otimes c}.$$

There is a notion of δ-semistability for sheaves with decorations of type (a,b,c) (depending on a polynomial δ). Moduli spaces of such sheaves form a sequence of flips similar to M. Thaddeus. One of the main problems is its termination. This follows from boundedness of the family of all δ-semistable sheaves with fixed decoration and Hilbert polynomial. Once this is checked the notion of semistability for large δ does not depend on δ and all the sheaves in the limit correspond to singular principal G-bundles. There is also an important issue if this gives a projective scheme in general but this can also be checked.

The difference if one wants to deal with arbitrary characteristic is that not every representation is a subrepresentation of the tensor representation ([**35**, Proposition 5.3] holds only in characteristic zero and the claim that it generalizes to positive characteristic is false). Anyway, every representation can be embedded into an appropriate tensor product of symmetric powers so one need to generalize the known moduli spaces of sheaves with decorations of type (a,b,c) to more general decorations. Once this is done the rest of the proof is similar although one cannot longer check that the moduli space of singular principal G-bundles is always projective.

5.5. Geometry of moduli spaces of G-bundles. In case of $\rho : G = \mathrm{SL}(V) \to \mathrm{GL}(V)$, the moduli space of semistable principal ρ-sheaves is just the usual moduli space of semistable torsion free sheaves with trivial determinant. But already the case of $G = \mathrm{PGL}(V)$ is very interesting, as in this case one gets moduli spaces of Azumaya algebras and one can hope that this can be used to study the Brauer group of a variety. In this case quasi-projective moduli spaces were also constructed by N. Hoffmann and U. Stuhler (see [**29**, Propostion 4.1]), using a different method motivated by the constuction of Brandt groupoid. M. Lieblich in [**44**] used twisted sheaves to construct a moduli stack of semistable PGL_n-bundles on surfaces if n does not divide the characteristic of the base field. He proved that it is an Artin quasi-proper stack of finite presentation and it satisfies properties similar to those of moduli spaces of semistable sheaves on surfaces.

Recently, the geometry of moduli spaces of principal G-bundles in the characteristic zero case was studied by V. Balaji in [**3**]. He constructed the Donaldson–Uhlenbeck compactification of the moduli space of stable principal G-bundles on surfaces. He also showed that for large "second Chern class" the moduli spaces are non-empty. In the vector bundle case much more is known due to the work of Donaldson, Gieseker–Li, O'Grady and many others in characteristic zero and of the author in the positive characteristic case (see [**41**] for precise references and history of the subject). Namely, the corresponding moduli spaces are not only non-empty but also irreducible and reduced varieties of expected dimension.

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