Erratum to "On boundedness of semistable sheaves"

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Unfortunately, the proof of [2, Theorem 3.4] contains a gap. The problem is that in the notation of the proof we have $\Delta(F)p^*\mathscr{O}_\Lambda(1)q^*H^{n-3} = \Delta(F_s)H_s^{n-3} \geq 0$. This proves that Bogomolov's inequality holds for the collection $q^*Hp^*\mathscr{O}_\Lambda(1)q^*H^{n-3}$ but it does not imply that it holds for the collection $p^*\mathscr{O}_\Lambda(1)q^*H^{n-2}$. Here we give a different proof of this theorem that is based on the idea coming from [1]. The same problem occurs in the proof of Theorem 3.13 and it can be repaired in the same way as below.

THEOREM 0.1. *Let H be a hyperplane on $\mathbb{P}^n$ and let E be a slope H-semistable torsion free coherent sheaf on $\mathbb{P}^n$. Then $\Delta(E)H^{n-2} \geq 0$.*

Proof. The proof is by induction on the dimension n and the rank r of E. We keep notation from the proof of [2, Theorem 3.4]. Let $0 = E_0 \subset E_1 \subset \cdots \subset E_m = q^*E$ be the Harder–Narasimhan filtration with respect to $p^*\mathscr{O}_\Lambda(1)q^*H^{n-2}$. If $m = 1$ then the restriction of E to a general hyperplane is semistable and the assertion follows by the induction assumption. So we can assume that $m > 1$. Let us set $F_i = E_i/E_{i-1}$, $r_i = \mathrm{rk}\, F_i$ and let μ_i be the slope of F_i with respect to $p^*\mathscr{O}_\Lambda(1)q^*H^{n-2}$. There exist integers a_i, b_i such that $c_1(F_i) = a_i q^*H + b_i N$, where N is the exceptional divisor of q. Then we have $r_i\mu_i = c_1(F_i)p^*\mathscr{O}_\Lambda(1)q^*H^{n-2} = a_i + b_i$. Our induction assumption implies that Bogomolov's inequality holds on Y for the collection q^*H^{n-1} for sheaves of rank $< r$. Since F_j is $(p^*\mathscr{O}_\Lambda(1), q^*H^{n-2})$-semistable, [2, Proposition 3.1] implies that $\Delta(F_j)q^*H^{n-2} \geq 0$. Therefore using $a_i = r_i\mu_i - b_i$

we get

$$\frac{\Delta(E)H^{n-2}}{r} = \sum \frac{\Delta(F_i)q^*H^{n-2}}{r_i} - \frac{1}{r}\sum_{i<j} r_i r_j \left(\frac{c_1(F_i)}{r_i} - \frac{c_1(F_j)}{r_j}\right)^2 q^*H^{n-2}$$
$$\geq \frac{1}{r}\sum_{i<j} r_i r_j \left(\left(\frac{b_i}{r_i} - \frac{b_j}{r_j}\right)^2 - \left(\frac{a_i}{r_i} - \frac{a_j}{r_j}\right)^2\right) = 2\sum b_i\mu_i - \frac{1}{r}\sum_{i<j} r_i r_j(\mu_i - \mu_j)^2.$$

On the other hand $q_*E_i \subset E$, so

$$\frac{\sum_{j\leq i} a_j}{\sum_{j\leq i} r_j} \leq \mu = \mu(E).$$

Hence $\sum_{j\leq i} b_j \geq \sum_{j\leq i} r_j(\mu_j - \mu)$ and we have

$$\sum b_i\mu_i = \sum_i (\sum_{j\leq i} b_j)(\mu_i - \mu_{i+1}) \geq \sum_i (\sum_{j\leq i} r_j(\mu_j - \mu))(\mu_i - \mu_{i+1}) = \sum r_i\mu_i^2 - r\mu^2 = \sum_{i<j} \frac{r_i r_j}{r}(\mu_i - \mu_j)^2.$$

Summing up, we get

$$\Delta(E)H^{n-2} \geq \sum_{i<j} \frac{r_i r_j}{r}(\mu_i - \mu_j)^2 \geq 0.$$

□

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References

[1] Langer, Adrian Semistable sheaves in positive characteristic. *Ann. of Math. (2)* **159** (2004), 251–276.

[2] Langer, Adrian On boundedness of semistable sheaves, *Doc. Math.* **27** (2022), 1–16.