MAGIT Exercises, Series 9

Exercise 1.

Let G and H be groups. Show that if k is a field then

$$H_n(G \times H; k) \simeq \bigoplus_{p+q=n} H_p(G; k) \otimes_k H_q(H; k).$$

Exercise 2.

Prove that $\operatorname{Ext}^1\left(\mathbb{Z}\left[\frac{1}{p}\right], \mathbb{Z}\right) \simeq \hat{\mathbb{Z}}_p/\mathbb{Z}$.

Exercise 3.

Let I be the poset $\{a, b, c\}$, $a < c$ and $b < c$. Let $\mathcal{A}$ be an abelian category and assume it is complete and the product of any set of surjections is a surjection. Let $F : \mathcal{A}^I \to \mathcal{A}$ be the functor from the category $\mathcal{A}^I$ of diagrams of shape I in $\mathcal{A}$ to $\mathcal{A}$, associating to a diagram its limit over I. Show that

1. if $X \in \mathcal{A}^I$ then $F(X)$ is the fiber product of $X(a)$ and $X(b)$ over $X(c)$,
2. $R^1F(X)$ is the cokernel of a certain map $X(a) \times X(b) \to X(c)$,
3. $R^nF(X) = 0$ for $n \neq 0, 1$.

Exercise 4.

Let p be a prime number and let $m = 2023p^{2023}$. Show that for all $n \geq 0$ we have

1. $\operatorname{Ext}^n_{\mathbb{Z}/m}(\mathbb{Z}/p\mathbb{Z}, \mathbb{Z}/p\mathbb{Z}) \simeq \mathbb{Z}/p\mathbb{Z}$.
2. $\operatorname{Ext}^n_{\mathbb{Z}/m}(\bigoplus_{i=1}^{\infty}(\mathbb{Z}/p\mathbb{Z}), \mathbb{Z}/p\mathbb{Z})$ is a free $(\mathbb{Z}/p\mathbb{Z})$-module with an uncountable basis.

Exercise 5

Let $\cdots \to C_3^* \to C_2^* \to C_1^*$ be an inverse system of cochain complexes of abelian groups. Assume that for all n the inverse systems $\cdots \to C_3^n \to C_2^n \to C_1^n$ satisfy the Mittag–Leffler condition and set $C^* = \varprojlim C_i^*$. Prove that for all n we have a short exact sequence

$$0 \to \varprojlim{}^1 H^{n-1}(C_i^*) \to H^n(C^*) \to \varprojlim H^n(C_i^*) \to 0.$$