MAGIT Exercises, Series 8

Exercise 1.

Let $\{G_n\}$ be a homological δ-functor from $\mathcal{A}$ to $\mathcal{B}$ and assume $\mathcal{A}$ has enough projectives. Assume that for all $n > 0$ we have $G_n(P) = 0$ for all projective P. Show that $G_n = L_n(G_0)$.

Exercise 2.

Let $\mathcal{A}$ be an abelian category and let X_* be a chain complex in $\mathcal{A}$ such that for any object Y of $\mathcal{A}$ the chain complex $\mathrm{Hom}_{\mathcal{A}}(Y, X_*)$ is exact. Prove that X is homotopic to 0. Show an example when $X \neq 0$.

Exercise 3.

Let $\mathcal{A}, \mathcal{B}, \mathcal{C}$ be abelian categories. Assume that $\mathcal{A}$ has sufficiently many projectives. Let $F : \mathcal{A} \to \mathcal{B}$ be a right exact functor and $G : \mathcal{B} \to \mathcal{C}$ an exact functor. Show that for all $n \geq 0$ we have $G \circ L_n F \simeq L_n(G \circ F)$.

Exercise 4.

Let $\mathcal{A}, \mathcal{B}, \mathcal{C}$ be abelian categories. Assume that $\mathcal{A}$ has sufficiently many projectives. Let $F : \mathcal{A} \to \mathcal{B}$ and $G : \mathcal{B} \to \mathcal{C}$ be right exact functors. Is it true that $G \circ L_1 F \simeq L_1(G \circ F)$?

Exercise 5

Let M be a $\mathbb{Z}$-module such that $\mathrm{Hom}(M, \mathbb{Z}) = \mathrm{Ext}^1(M, \mathbb{Z}) = 0$. Show that $M = 0$. Use the following steps:

1. Show M is torsion-free.
2. Show that M is divisible.
3. Show that $M \simeq \mathbb{Q} \otimes_{\mathbb{Z}} M$.
4. If $M \neq 0$ then $M = \mathbb{Q} \oplus M'$ for some M'. But then we have $\mathrm{Ext}^1(\mathbb{Q}, \mathbb{Z}) = 0$, a contradiction.