MAGIT Homework, Series 7

Exercise 1.

Let $F : \mathcal{C} \to \mathcal{D}$ be a right exact functor between good abelian categories. An object M of $\mathcal{C}$ is F-*acyclic*, if $L_n F(M) = 0$ for all $n > 0$. Assume that the sequence

$$0 \to M_m \to P_m \to P_{m-1} \to \ldots \to P_0 \to A \to 0$$

is exact and all P_j are F-acyclic. Prove that

1. $L_i F(A) \simeq L_{i-m-1} F(M_m)$ for $i \geq m+2$,
2. $L_{m+1} F(A) \simeq \ker(F(M_m) \to F(P_m))$.

Exercise 2.

Use the previous exercise to show that if $P_\bullet \to A \to 0$ is a left resolution of A and all P_i are F-acyclic, then

$$L_i F(A) = H_i(F(P_\bullet)).$$

Exercise 3.

Let $\{F_n\}$ and $\{G_n\}$ be homological δ-functors from $\mathcal{A}$ to $\mathcal{B}$ and assume $\mathcal{A}$ has enough projectives. Let $\varphi : F_0 \to G_0$ be a natural transformation. Assume that for all $n > 0$ we have $G_n(P) = 0$ for all projective P. Show that there exists exactly one sequence of natural transformations $\{\varphi_n : F_n \to G_n\}$ starting with $\varphi_0 = \varphi$ such that for every short exact sequence in $\mathcal{A}$ the sequence $\{\varphi_n\}$ induces a map between long exact sequences associated to $\{F_n\}$ and $\{G_n\}$.

Exercise 4.

Show that if A and B are finite abelian groups then $Tor_m^{\mathbb{Z}}(A, B) = 0$ for $m > 1$ and $Tor_1^{\mathbb{Z}}(A, B) = A \otimes_{\mathbb{Z}} B$.

Exercise 5.

Let (X^*, d^*) be a cochain complex of R-modules. Assume $f : Y \to X^1$ and $g : X^2 \to Z$ are homomorphisms of R-modules and $gd^1 f = 0$. Show that the sequence

$$\ker d^1 f \to H^1(X^*) \to \ker gd^1/(\operatorname{Im} d^0 + \operatorname{Im} f) \to (\ker d^2 \cap \ker g)/\operatorname{Im} d^1 f \to H^2(X^*) \to Z/\operatorname{Im} gd^1$$

is exact.