MAGIT Exercises, Series 6

Exercise 1.

Let k be a field and let $R = k[x]/(x^2)$. The category of R-modules is equivalent to the category of k-vector spaces V together with an endomorphism $\varphi : V \to V$ such that $\varphi^2 = 0$. In particular, if $\dim_k V = 1$ then $\varphi = 0$. Compute $\mathrm{Ext}^1_R(k, k)$.

Exercise 2.

Let A be a torsion $\mathbb{Z}$-module. Show that

$$\mathrm{Ext}^1_{\mathbb{Z}}(A, \mathbb{Z}) = \mathrm{Hom}_{\mathbb{Z}}(A, U(1)),$$

where $U(1) = \{z \in \mathbb{C} : |z| = 1\}$.

Exercise 3.

Let $G = \mathbb{Z}/n\mathbb{Z}$. Let I_G be the augmentation ideal and let σ be the generator of G. Show that the map

$$\mathbb{Z}[G] \to I_G, \quad r \to r(1 - \sigma)$$

has the kernel $\mathbb{Z}[G](1 + \sigma + ... + \sigma^{n-1})$. Use this to construct a projective resolution of $\mathbb{Z}$ and to compute $H^i(G, M)$ for all G-modules M.

Exercise 4.

Let A be a $\mathbb{Z}$-module and let A_t be the submodule of torsion elements in A. Use only A_t to compute $Tor^{\mathbb{Z}}_m(A, \mathbb{Q}/\mathbb{Z})$ for all $m > 0$.

Exercise 5.

Let R be a ring (possibly non-commutative) and let I be a right ideal and J a left ideal in R. Show that $\mathrm{Tor}^R_1(R/I, R/J) \simeq (I \cap J)/IJ$.