MAGIT Exercises, Series 3

Exercise 1.

Show that if $\mathcal{D}$ is an abelian category, then $Funct(\mathcal{C}, \mathcal{D})$ is abelian.

Exercise 2.

Let $\mathcal{A}$ be a preadditive category and let $X, Y \in Ob\,\mathcal{A}$. Show that

1. $f : X \to Y$ is a monomorphism if and only if for every non-zero $g : Z \to X$ we have $fg \neq 0$.
2. $f : X \to Y$ is an epimorphism if and only if for every non-zero $g : Y \to Z$ we have $gf \neq 0$.

Exercise 3.

Let Rings be the category of possibly noncommutative rings with 1. Check which of the following functors are representable.

- Rings $\to$ Set sending a ring R to the set $\{r \in R : r^{2023} = 1\}$.
- Rings $\to$ Set sending a ring R to the set of nilpotent elements in R (i.e., to $\{r \in R : \exists_{n \geq 1} r^n = 0\}$).
- Rings $\to$ Set sending a ring R to the set $\{r^{2023} : r \in R\}$.

Exercise 4.

Let $\mathcal{A}$ be an abelian category. Prove that

1. the equalizer of morphisms $f, g : X \to Y$ in $\mathcal{A}$ coincides with the kernel of $f - g$.
2. the coequalizer of morphisms $f, g : X \to Y$ in $\mathcal{A}$ coincides with the cokernel of $f - g$.

Exercise 5.

Let Ab be the category of abelian groups. Show an example of a morphism $f : X \to Y$ in Ab such that the functor $Z \to \text{coker}(Hom(Z, X) \to Hom(Z, Y))$ is not representable and find coker f. Is the above functor corepresentable?