MAGIT Exercises, Series 11

Let $H^i(X, \mathcal{F})$ denote the right derived functor of the functor $\Gamma(X, \cdot)$ evaluated at the sheaf of abelian groups $\mathcal{F}$.

Exercise 1.

Let X be the set $\mathbb{Z}_{\geq 0}$ with the topology in which open subsets are $\emptyset$, $\mathbb{Z}_{\geq 0}$ and all sets $\{0, 1, ..., n\}$ for $n \geq 0$. Let $\mathcal{F}$ be a sheaf of abelian groups on X. Compute all cohomology groups $H^j(X, \mathcal{F})$.

Exercise 2.

Let $\mathcal{F}$ and $\mathcal{G}$ be sheaves of $\mathcal{R}_X$-modules. Show that for a point $x \in X$ the canonical map

$$(\mathcal{H}\mathrm{om}_{\mathcal{R}_X}(\mathcal{F}, \mathcal{G}))_x \to \mathcal{H}\mathrm{om}_{\mathcal{R}_x}(\mathcal{F}_x, \mathcal{G}_x)$$

need not be neither injective nor surjective. Do we have an isomorphism

$$(\mathcal{F} \otimes_{\mathcal{R}_X} \mathcal{G})_x \simeq \mathcal{F}_x \otimes_{\mathcal{R}_x} \mathcal{G}_x?$$

Exercise 3.

Let k be a field and let X be a topological space. Let k_X be the sheafification of the constant presheaf $U \to k$. Let $\mathcal{M}$ be a sheaf of k_X-modules. Prove that the following conditions are equivalent:

1. For any open subset $U \subset X$ the restriction morphism $\mathcal{M}(X) \to \mathcal{M}(U)$ is surjective.
2. $\mathcal{M}$ is injective.

Hint: try to prove an analogue of Baer's criterion for sheaves of rings/fields.

Exercise 4.

Let X be a topological space and let E be a presheaf. A *subpresheaf* of E is a presheaf F such that $F(U) \subset E(U)$ for all open $U \subset X$ and such that for all open subsets $U \subset V \subset X$ the restriction maps $F(U) \to F(V)$ are induced by the restriction maps for E. Suppose that E is a sheaf. Show that a subpresheaf F of E is a sheaf if and only if for every open subset $U \subset X$ and for every open covering $\{U_i\}_{i \in I}$ of U and for every $s \in E(U)$ with $s|_{U_i} \in F(U_i)$ we have $s \in F(U)$.

Exercise 5.

Let $\mathcal{F}$ be a sheaf of abelian groups on a topological space X. For a closed subset $j : Y \hookrightarrow X$ we set $\Gamma(Z, \mathcal{F}) := \Gamma(Z, j^{-1}\mathcal{F})$. Let $Y_1, Y_2 \subset X$ be closed subsets. Show that the sequence

$$0 \to \Gamma(Y_1 \cup Y_2, \mathcal{F}) \to \Gamma(Y_1, \mathcal{F}) \oplus \Gamma(Y_2, \mathcal{F}) \to \Gamma(Y_1 \cap Y_2, \mathcal{F})$$

is exact. Is this sequence exact also on the right?