GA Series 2

Ex. 1.

Let A be a commutative ring. Show that for every $x \in X = \operatorname{Spec} A$ we have

$$\mathcal{O}_{X,x} \simeq A_{\mathfrak{p}_x},$$

where $\mathfrak{p}_x$ is the prime ideal corresponding to x.

Ex. 2.

Let k be algebraically closed and let R_1 and R_2 be reduced finitely generated k-algebras. Show that $R_1 \otimes_k R_2$ is reduced. Is it true if we don't assume that R_1 and R_2 are finitely generated? Is it true if k is not algebraically closed?

Ex. 3.

Show that an open subset $U \subset \operatorname{Spec} A$ is quasi-compact if and only if $Y = \operatorname{Spec} A - U$ is of the form $V(I)$ for some finitely generated I. When is U compact?

Ex. 4.

Let A be a local ring. Show that $\operatorname{Spec} A$ is connected.

Ex. 5.

A commutative ring A is *Boole* if for every $a \in A$ we have $a^2 = a$. A topological space X is *totally disconnected*, if every connected component of X has exactly 1 point. Show that the functor Spec induces an equivalence between the category opposite to the category of Boole rings and the category of compact totally disconnected topological spaces.