

Elements of spectral theory in Banach algebras

1. Banach algebras, spectrum and spectral radius

Definition 1.1. A *Banach algebra* over $\mathbb{C}$ is a Banach space $\mathcal{A}$ equipped with a multiplication

$$\mathcal{A} \times \mathcal{A} \ni (a, b) \mapsto ab \in \mathcal{A},$$

which satisfies $a(\lambda b) = (\lambda a)b = \lambda(ab)$, $a(bc) = (ab)c$, $a(b+c) = ab+ac$ and $(b+c)a = ba+ca$ for all $a, b, c \in \mathcal{A}$ and $\lambda \in \mathbb{C}$. Furthermore, the multiplication is required to be compatible with the norm in the sense that

$$(1.1) \quad \|ab\| \leq C\|a\|\|b\| \quad (\text{in what follows, we will assume } C = 1).$$

Remark 1.2. (i) By induction, the submultiplicativity of the norm gives, for $a_1, a_2, \dots, a_n \in \mathcal{A}$,

$$\|a_1 a_2 \dots a_n\| \leq \|a_1\| \|a_2\| \dots \|a_n\|, \quad n \geq 1.$$

(ii) We will always assume that a Banach algebra is unital, i.e., it has an identity element e satisfying $\|e\| = 1$. In general, $\mathcal{A}$ need not have an identity, but one can adjoin an identity by passing to the slightly larger unital algebra $\mathcal{A} \oplus \mathbb{C}$, with multiplication $(a + \lambda 1)(b + \mu 1) = ab + \lambda b + \mu a + \lambda \mu 1$ and norm $\|a + \lambda 1\| = \|a\| + |\lambda|$.

Example 1.3. (i) If K is a non-empty compact set, then the space $C(K)$ of continuous, complex valued functions on K , with the pointwise multiplication and the supremum norm, is a Banach algebra.

(ii) For a given integer n , the complex space $L^1(\mathbb{R}^n)$ with the multiplication given by the convolution

$$f * g(x) = \int_{\mathbb{R}^n} f(y)g(x-y)dy, \quad x \in \mathbb{R}^n,$$

forms a Banach algebra.

(iii) Another example is the space $\mathcal{B}(X)$ of all bounded linear operators from a Banach space X into itself, with the multiplication given by the composition

$$(S, T) \mapsto ST,$$

and the norm equal to the usual operator norm

$$\|T\|_{\mathcal{B}(X)} = \sup_{\|v\| \leq 1} \|Tv\| = \sup_{v \neq 0} \frac{\|Tv\|}{\|v\|}.$$

(iv) An important special case of (iii) is $\mathcal{A} = \mathcal{B}(H)$, where H is a Hilbert space. In this case, $\mathcal{B}(H)$ carries an additional structure in the form of the *adjoint operation* $T \mapsto T^*$ defined by

$$\langle Tv, w \rangle = \langle v, T^*w \rangle, \quad v, w \in H.$$

The adjoint satisfies

$$(S + T)^* = S^* + T^*, \quad (\alpha T)^* = \bar{\alpha} T^*, \quad (ST)^* = T^* S^*, \quad (T^*)^* = T.$$

Moreover, we have $\|T^*T\| = \|T\|^2$. Any complex Banach algebra which admits the adjoint operation enjoying the above conditions is called a *C^* -algebra*.

Definition 1.4. An element $a \in \mathcal{A}$ is called *invertible*, if there exists $b \in \mathcal{A}$ such that $ab = ba = e$. The set of all invertible elements forms a group under multiplication (in particular, the inverse element is unique).

Lemma 1.5. Let $\mathcal{A}$ be a Banach algebra. Then the set of invertible elements is open in $\mathcal{A}$. More precisely, for every invertible $a \in \mathcal{A}$ and any $b \in \mathcal{A}$ with $\|b\| < \|a^{-1}\|^{-1}$, the element $a - b$ is invertible and

$$(a - b)^{-1} = a^{-1} \sum_{k=0}^{\infty} (ba^{-1})^k.$$

Proof. If $\|b\| < \|a^{-1}\|^{-1}$, then $\|ba^{-1}\| \leq \|b\| \|a^{-1}\| < 1$ and hence the series $\sum_{k=0}^{\infty} (ba^{-1})^k$ converges absolutely in $\mathcal{A}$. It remains to note that

$$(a-b)a^{-1} \sum_{k=0}^N (ba^{-1})^k = e - (ba^{-1})^{N+1} \xrightarrow{N \rightarrow \infty} e, \quad a^{-1} \sum_{k=0}^N (ba^{-1})^k (a-b) = e - (a^{-1}b)^{N+1} \xrightarrow{N \rightarrow \infty} e,$$

which implies that $a^{-1} \sum_{k=0}^{\infty} (ba^{-1})^k$ is the left and right inverse of $a-b$, thus equals $(a-b)^{-1}$. $\square$

Corollary 1.6. *The map $a \mapsto a^{-1}$ is continuous on the set of invertible elements.*

Proof. Suppose that $(a_n)_{n \geq 1}$ is a sequence of invertible elements of $\mathcal{A}$ which converges to an invertible a . Then $\|a_n - a\| < \|a^{-1}\|^{-1}$ for sufficiently large n , and hence, by the previous lemma,

$$a_n^{-1} = (a - (a - a_n))^{-1} = a^{-1} \sum_{k=0}^{\infty} ((a - a_n)a^{-1})^k = a^{-1} + \sum_{k=1}^{\infty} ((a - a_n)a^{-1})^k.$$

This implies

$$\|a_n^{-1} - a^{-1}\| = \left\| \sum_{k=1}^{\infty} ((a - a_n)a^{-1})^k \right\| \leq \sum_{k=1}^{\infty} \|(a - a_n)a^{-1}\|^k \leq \sum_{k=1}^{\infty} \|a - a_n\|^k \|a^{-1}\|^k = \frac{\|a - a_n\| \|a^{-1}\|}{1 - \|a - a_n\| \|a^{-1}\|}$$

and hence $a_n^{-1} \rightarrow a^{-1}$. $\square$

Definition 1.7. Let $\mathcal{A}$ be a unital Banach algebra. The *spectrum* of $x \in \mathcal{A}$ is the set

$$\sigma(x) = \{\lambda \in \mathbb{C} : \lambda e - x \text{ is not invertible in } \mathcal{A}\}.$$

For $\lambda \in \mathbb{C} \setminus \sigma(x)$, the element $R(\lambda) = (\lambda e - x)^{-1}$ is called the *resolvent* of x at λ .

To study some basic properties of $\sigma(x)$, we need a concept of a Banach-space-valued holomorphic function.

Definition 1.8. Let X be a complex Banach space and let Ω be an open subset of $\mathbb{C}$. A continuous function $f : \Omega \rightarrow X$ is called *analytic*, if for any functional $\varphi \in X^*$ the composition $\varphi \circ f : \Omega \rightarrow \mathbb{C}$ is analytic.

Theorem 1.9. *A continuous function $f : \Omega \rightarrow X$ is analytic if and only if for any $z \in \Omega$ the limit $\lim_{h \rightarrow 0} (f(z+h) - f(z))/h$ exists and is finite.*

Proof. Left as an exercise. $\square$

Theorem 1.10. *Let $\mathcal{A}$ be a complex unital Banach algebra. Then for any $x \in \mathcal{A}$, the spectrum $\sigma(x)$ is a non-empty compact subset of $\mathbb{C}$.*

Proof. By Lemma 1.5, if $|\lambda| > \|x\|$, then $\lambda e - x$ is invertible; thus $\sigma(x) \subset B(0, \|x\|)$. Next, for $|\lambda| > 2\|x\|$,

$$(1.2) \quad \|(\lambda e - x)^{-1}\| = |\lambda|^{-1} \left\| \sum_{k=0}^{\infty} \lambda^{-k} x^k \right\| \leq \frac{2}{|\lambda|}.$$

To show that $\sigma(x)$ is closed, note that if $\lambda \notin \sigma(x)$, then $\lambda e - x$ is invertible. But the set of invertible elements in $\mathcal{A}$ is open, so $\mu e - x$ is invertible for μ sufficiently close to λ .

Finally, suppose that $\sigma(x) = \emptyset$. Then the resolvent function $R(\lambda) = (\lambda e - x)^{-1} : \mathbb{C} \rightarrow \mathcal{A}$ is differentiable:

$$\begin{aligned} \frac{R(\lambda+h) - R(\lambda)}{h} &= \frac{((\lambda+h)e - x)^{-1} - (\lambda e - x)^{-1}}{h} \\ &= \frac{((\lambda+h)e - x)^{-1}(\lambda e - x)(\lambda e - x)^{-1} - ((\lambda+h)e - x)^{-1}((\lambda+h)e - x)(\lambda e - x)^{-1}}{h} \\ &= -((\lambda+h)e - x)^{-1}(\lambda e - x)^{-1} \xrightarrow{h \rightarrow 0} -[(\lambda e - x)^{-1}]^2. \end{aligned}$$

Furthermore, by (1.2), the resolvent R is bounded outside $B(0, 2\|x\|)$ (and hence bounded globally, by continuity). Consequently, for any $\varphi \in \mathcal{A}^*$, the complex-valued function $\lambda \mapsto \varphi(R(\lambda))$ is bounded and analytic on $\mathbb{C}$, and hence it is constant, in the light of Liouville's theorem. But by (1.2) again, we have $R(\lambda) \rightarrow 0$ as $\lambda \rightarrow \infty$. Combining this with the preceding observations, we obtain that $\varphi(R(\lambda)) = 0$ for all $\varphi \in \mathcal{A}^*$ and hence $R(\lambda) = 0$ for all $\lambda \in \mathbb{C}$, a contradiction. $\square$

It is natural to ask whether the spectra of two elements $x, y \in \mathcal{A}$ are close, in some suitably defined sense, if x and y are close to each other. The next theorem, left as an exercise, gives a very simple answer.

Theorem 1.11. *Suppose A is a Banach algebra, $x \in A$, Ω is an open set in $\mathbb{C}$, and $\sigma(x) \subset \Omega$. Then there exists $\delta > 0$ such that $\sigma(x + y) \subset \Omega$ for every $y \in A$ satisfying $\|y\| < \delta$.*

Definition 1.12. The *spectral radius* of $x \in \mathcal{A}$ is defined by $r(x) = \sup \{ |\lambda| : \lambda \in \sigma(x) \}$.

Theorem 1.13. *Let $x \in \mathcal{A}$. Then $r(x) = \lim_{n \rightarrow \infty} \|x^n\|^{1/n} = \inf_{n \geq 1} \|x^n\|^{1/n}$.*

In the proof, we will need the following auxiliary observation.

Lemma 1.14. *Let $x \in \mathcal{A}$. Then the sequence $(\|x^n\|^{1/n})_{n \geq 1}$ is convergent and*

$$\lim_{n \rightarrow \infty} \|x^n\|^{1/n} = \inf_{n \geq 1} \|x^n\|^{1/n}.$$

Proof. If $x^n = 0$ for some $n \geq 1$, then $x^m = 0$ for all $m \geq n$ and the claim holds. Thus, assume that $x^n \neq 0$ for every $n \geq 1$ and set $u_n = \log \|x^n\|$. We have $\|x^{n+p}\| \leq \|x^n\| \|x^p\|$ and hence $u_{n+p} \leq u_n + u_p$ for all $n, p \geq 1$. Now fix $p \geq 1$ and, for every $n \geq 1$, write

$$n = qp + r, \quad q \geq 0, \quad 0 \leq r < p.$$

By the repeated application of the preceding inequality for $(u_n)_{n \geq 1}$, we get $u_n \leq qu_p + u_r$ and therefore

$$\frac{u_n}{n} \leq \frac{u_p}{p} + \frac{u_r}{n}.$$

Since $r < p$ and p is fixed, the second term tends to zero as $n \rightarrow \infty$ and we obtain $\limsup_{n \rightarrow \infty} \frac{u_n}{n} \leq \frac{u_p}{p}$. But p was arbitrary, so we get $\limsup_{n \rightarrow \infty} \frac{u_n}{n} \leq \inf_{p \geq 1} \frac{u_p}{p} \leq \liminf_{n \rightarrow \infty} \frac{u_n}{n}$. Thus the limit exists and $\lim_{n \rightarrow \infty} \frac{u_n}{n} = \inf_{n \geq 1} \frac{u_n}{n}$. Exponentiating gives the claim. $\square$

Proof of Theorem 1.13. We may and do assume that $x \neq 0$. Pick an arbitrary $\lambda \in \sigma(x)$. For any integer n we have

$$\lambda^n e - x^n = (\lambda e - x)((\lambda e)^{n-1} + (\lambda e)^{n-2}x + \dots + x^{n-1}) = ((\lambda e)^{n-1} + (\lambda e)^{n-2}x + \dots + x^{n-1})(\lambda e - x).$$

Therefore $\lambda^n \in \sigma(x^n)$: indeed, if $\lambda^n \notin \sigma(x^n)$, then the double inequality above implies the existence of the left and right inverse to $\lambda e - x$ (as one easily shows, then $\lambda e - x$ is invertible). Thus $|\lambda^n| \leq \|x^n\|$ and hence $r(x) \leq \inf_{n \geq 1} \|x^n\|^{1/n}$, since λ and n were chosen arbitrarily.

It remains to prove the reverse inequality. By the calculation from the proof of Theorem 1.10, the resolvent

$$R(\lambda) = (\lambda e - x)^{-1} = \lambda^{-1} \sum_{n=0}^{\infty} (\lambda^{-1} x)^n$$

is analytic on the set $\{\lambda \in \mathbb{C} : |\lambda| > r(x)\}$ and vanishes at infinity. Consequently, the function

$$f(\lambda) = \lambda \sum_{n=0}^{\infty} (\lambda x)^n = \begin{cases} 0 & \text{if } \lambda = 0, \\ R(1/\lambda) & \text{if } 0 < |\lambda| < (r(x))^{-1} \end{cases}$$

is holomorphic on the ball $B(0, r(x)^{-1})$. But the radius of convergence of the above power series equals $(\limsup_{n \rightarrow \infty} \|x^n\|^{1/n})^{-1}$; thus $r(x)^{-1} \leq (\limsup_{n \rightarrow \infty} \|x^n\|^{1/n})^{-1}$, which is the desired reverse bound. $\square$

2. Holomorphic functional calculus

Suppose that x is an element of a Banach algebra A , and $f : \mathbb{C} \rightarrow \mathbb{C}$ is a given function. Can we meaningfully define the element $f(x) \in \mathcal{A}$? For some choices of f this is clear: for instance, if $f(\lambda) = a_0 + a_1\lambda + \dots + a_n\lambda^n$ is a polynomial, the only reasonable choice seems to be

$$f(x) = a_0 e + a_1 x + \dots + a_n x^n.$$

Below, we will answer the above question in the case when f is a holomorphic function on an open subset $\Omega \subset \mathbb{C}$ containing $\sigma(x)$. We will exploit the integration of vector-valued functions.

Definition 1.15. Let X be a Banach space and let $f : [a, b] \rightarrow X$ be a continuous function. We define $\int_a^b f(t)dt$ as the Riemann integral.

Suppose K is a compact subset of an open set $\Omega \subset \mathbb{C}$, and Γ is a collection of finitely many oriented line intervals $\gamma_1, \gamma_2, \dots, \gamma_N$ in $\mathbb{C}$, none of which intersects K . It is well known that Γ can be chosen so that

$$\text{Ind}_\Gamma(z) = \frac{1}{2\pi i} \int_\Gamma \frac{d\zeta}{\zeta - z} = \begin{cases} 1, & z \in K, \\ 0, & z \notin \Omega. \end{cases}$$

In such a case, we will say that a contour Γ surrounds K in Ω . Our starting point is the following fact.

Lemma 1.16. Suppose A is a Banach algebra, $x \in A$, $\lambda \notin \sigma(x)$ and Γ surrounds $\sigma(x)$ in $\mathbb{C} \setminus \{\lambda\}$. Then

$$\frac{1}{2\pi i} \int_\Gamma (\lambda - \zeta)^n (\zeta e - x)^{-1} d\zeta = (\lambda e - x)^n \quad \text{for } n \in \mathbb{Z}.$$

Proof. Denote the integral by a_n . If $\zeta \notin \sigma(x)$, we have $(\zeta e - x)^{-1} = (\lambda e - x)^{-1} + (\lambda - \zeta)(\lambda e - x)^{-1}(\zeta e - x)^{-1}$ and therefore a_n is the sum of

$$\frac{1}{2\pi i} (\lambda e - x)^{-1} \int_\Gamma (\lambda - \zeta)^n d\zeta = 0 \quad \text{and} \quad (\lambda e - x)^{-1} \frac{1}{2\pi i} \int_\Gamma (\lambda - \zeta)^{n+1} (\zeta e - x)^{-1} d\zeta$$

(the first integral vanishes, since $\text{Ind}_\Gamma(\lambda) = 0$). This implies $(\lambda e - x)a_n = a_{n+1}$ and hence it is enough to establish the case $n = 0$, i.e.,

$$(1.3) \quad \frac{1}{2\pi i} \int_\Gamma (\zeta e - x)^{-1} d\zeta = e.$$

Let Γ_1 be a positively oriented circle centered at 0 and radius $r > \|x\|$. On Γ_1 , we have $(\zeta e - x)^{-1} = \sum_{k=0}^{\infty} \zeta^{-k-1} x^k$ and the term-wise integration of this series gives (1.3), with Γ_1 in place of Γ . Since the integrand in (1.3) is a holomorphic function in the complement of $\sigma(x)$ (see the proof of Theorem 1.10) and $\text{Ind}_{\Gamma_1}(z) = 1 = \text{Ind}_\Gamma(z)$ for every $z \in \sigma(x)$, the integral (1.3) is unaffected if Γ is replaced by Γ_1 . $\square$

Corollary 1.17. Let $R(\lambda) = P(\lambda) + \sum_{m,k} a_{m,k}(\lambda - c_m)^{-k}$ be a rational function with poles at the points c_m (here P is a polynomial, and the sum has only finitely many terms). If $x \in A$ and $\sigma(x)$ contains no pole of R , define

$$(1.4) \quad R(x) = P(x) + \sum_{m,k} a_{m,k}(x - c_m e)^{-k}.$$

If Ω is an open set in $\mathbb{C}$ that contains $\sigma(x)$ and in which R is holomorphic, and Γ surrounds $\sigma(x)$ in Ω , then

$$(1.5) \quad R(x) = \frac{1}{2\pi i} \int_\Gamma R(\zeta)(\zeta e - x)^{-1} d\zeta.$$

Note that (1.4) is certainly the most natural definition of a rational function of $x \in A$. The conclusion (1.5) shows that the Cauchy formula gives the same result. This motivates the following definition.

Definition 1.18. Suppose A is a Banach algebra, Ω is an open set in $\mathbb{C}$, and let $H(\Omega)$ denote the (commutative) algebra of all complex-valued holomorphic functions on Ω . By Theorem 1.11, $A_\Omega = \{x \in A : \sigma(x) \subset \Omega\}$ is an open subset of A . We define $H(A_\Omega)$ to be the set of all A -valued functions $\tilde{f}$ with domain A_Ω that arise from a function $f \in H(\Omega)$ by the formula

$$\tilde{f}(x) = \frac{1}{2\pi i} \int_\Gamma f(\zeta)(\zeta e - x)^{-1} d\zeta,$$

where Γ is any contour that surrounds $\sigma(x)$ in Ω .

Theorem 1.19. The space $H(A_\Omega)$ is a complex algebra and the mapping $f \mapsto \tilde{f}$ is an algebra isomorphism of $H(\Omega)$ onto $H(A_\Omega)$ which is continuous in the following sense: if $f_1, f_2, \dots \in H(\Omega)$ and $f_n \rightarrow f$ uniformly on compact subsets of Ω , then

$$\tilde{f}_n(x) \rightarrow \tilde{f}(x) \quad (x \in A_\Omega).$$

Finally, if $u(\lambda) = \lambda$ and $v(\lambda) = 1$ in Ω , then $\tilde{u}(x) = x$ and $\tilde{v}(x) = e$ for every $x \in A_\Omega$.

Proof. The last assertion follows from Corollary 1.17. The linearity of the mapping $f \mapsto \tilde{f}$ is evident. If $\tilde{f} = 0$, then we pick $\lambda \in \Omega$, note that $x = \lambda e \in A_\Omega$ and hence $f(\lambda)e = \tilde{f}(x) = 0$; thus $f = 0$ and hence $f \mapsto \tilde{f}$ is one-to-one. To show continuity, note that $\|(\zeta e - x)^{-1}\|$ is bounded on Γ and hence

$$\|\tilde{f}_n(x) - \tilde{f}(x)\| \leq \sup_{\zeta \in \Gamma} \|(\zeta e - x)^{-1}\| \int_{\Gamma} |f_n(\zeta) - f(\zeta)| d|\zeta| \xrightarrow{n \rightarrow \infty} 0.$$

It remains to prove that $f \mapsto \tilde{f}$ is multiplicative: if $f, g \in H(\Omega)$ and $h = fg$, we must show that $\tilde{h}(x) = \tilde{f}(x)\tilde{g}(x)$ for $x \in A_\Omega$. Now if f and g are rational functions without poles in Ω , then this follows directly from Corollary 1.17. In the general case, we apply Runge's theorem, which allows us to approximate f and g by rational functions f_n and g_n , uniformly on compact subsets of Ω . Then $f_n g_n \rightarrow h$ uniformly on compact subsets of Ω , and then assertion follows from the continuity of the mapping $f \rightarrow \tilde{f}$. $\square$

Theorem 1.20. *Suppose $x \in A_\Omega$ and $f \in H(\Omega)$. Then*

- (i) *the element $\tilde{f}(x)$ is invertible in A if and only if $f(\lambda) \neq 0$ for every $\lambda \in \sigma(x)$;*
- (ii) *(the spectral mapping theorem) we have $\sigma(\tilde{f}(x)) = f(\sigma(x))$.*

Proof. (i) If f has no zero on $\sigma(x)$, then $g = \frac{1}{f}$ is holomorphic in an open set Ω_1 such that $\sigma(x) \subset \Omega_1 \subset \Omega$. Since $fg = 1$ in Ω_1 , the previous theorem (with Ω_1 in place of Ω) shows that

$$\tilde{f}(x)\tilde{g}(x) = e,$$

and thus $\tilde{f}(x)$ is invertible. Conversely, if $f(\alpha) = 0$ for some $\alpha \in \sigma(x)$, then there exists $h \in H(\Omega)$ such that $(\lambda - \alpha)h(\lambda) = f(\lambda)$ for $\lambda \in \Omega$. By the previous theorem,

$$(x - \alpha e)\tilde{h}(x) = \tilde{f}(x) = \tilde{h}(x)(x - \alpha e).$$

Since $x - \alpha e$ is not invertible in A , neither is $\tilde{f}(x)$.

(ii) Fix $\beta \in \mathbb{C}$. By definition, $\beta \in \sigma(\tilde{f}(x))$ if and only if $\tilde{f}(x) - \beta e$ is not invertible in A . By part (i), applied to $f - \beta$ in place of f , this happens if and only if $f - \beta$ has a zero in $\sigma(x)$. That is, $f(\lambda) = \beta$ for some $\lambda \in \sigma(x)$, which is equivalent to $\beta \in f(\sigma(x))$. Therefore, $\sigma(\tilde{f}(x)) = f(\sigma(x))$. $\square$

The spectral mapping theorem makes it possible to include composition of functions among the operations of the symbolic calculus.

Theorem 1.21. *Suppose $x \in A_\Omega$, $f \in H(\Omega)$, Ω_1 is an open set containing $f(\sigma(x))$, $g \in H(\Omega_1)$, and $h(\lambda) = g(f(\lambda))$ in $\Omega_0 = \{\lambda \in \Omega : f(\lambda) \in \Omega_1\}$. Then $\tilde{f}(x) \in A_{\Omega_1}$ and $\tilde{h}(x) = \tilde{g}(\tilde{f}(x))$. That is,*

$$\tilde{h} = \tilde{g} \circ \tilde{f} \quad \text{if} \quad h = g \circ f.$$

Proof. By part (ii) of the previous theorem, $\sigma(\tilde{f}(x)) \subset \Omega_1$, and therefore $\tilde{g}(\tilde{f}(x))$ is well-defined. Fix a contour Γ_1 that surrounds $f(\sigma(x))$ in Ω_1 . There is an open set W such that $\sigma(x) \subset W \subset \Omega_0$, small enough so that $\text{Ind}_{\Gamma_1}(f(\lambda)) = 1$ for $\lambda \in W$. Next, fix a contour Γ_0 that surrounds $\sigma(x)$ in W . If $\zeta \in \Gamma_1$, then $\frac{1}{\zeta - f} \in H(W)$. Hence Theorem 1.19, with W in place of Ω , shows that

$$(\zeta e - \tilde{f}(x))^{-1} = \frac{1}{2\pi i} \int_{\Gamma_0} (\zeta - f(\lambda))^{-1} (\lambda e - x)^{-1} d\lambda.$$

Since Γ_1 surrounds $\sigma(\tilde{f}(x))$ in Ω_1 , the definition of the functional calculus gives

$$\begin{aligned} g(\tilde{f}(x)) &= \frac{1}{2\pi i} \int_{\Gamma_1} g(\zeta)(\zeta e - \tilde{f}(x))^{-1} d\zeta = \frac{1}{2\pi i} \int_{\Gamma_1} g(\zeta) \left[\frac{1}{2\pi i} \int_{\Gamma_0} (\zeta - f(\lambda))^{-1} (\lambda e - x)^{-1} d\lambda \right] d\zeta \\ &= \frac{1}{2\pi i} \int_{\Gamma_0} \left[\frac{1}{2\pi i} \int_{\Gamma_1} \frac{g(\zeta)}{\zeta - f(\lambda)} d\zeta \right] (\lambda e - x)^{-1} d\lambda. \end{aligned}$$

By the Cauchy integral formula and the equality $\text{Ind}_{\Gamma_1}(f(\lambda)) = 1$ for $\lambda \in W$, we see that the integral in the square brackets is $g(f(\lambda))$. Consequently,

$$g(\tilde{f}(x)) = \frac{1}{2\pi i} \int_{\Gamma_0} g(f(\lambda))(\lambda e - x)^{-1} d\lambda = \frac{1}{2\pi i} \int_{\Gamma_0} h(\lambda)(\lambda e - x)^{-1} d\lambda = \tilde{h}(x). \quad \square$$

3. Exercises

In all exercises below, $\mathcal{A}$ denotes a unital Banach algebra.

1. Let $a, b \in \mathcal{A}$. Use the identity $(ab)^n = a(ba)^{n-1}b$ to prove that ab and ba always have the same spectral radius.

2. Let $x, y \in \mathcal{A}$.

(i) Prove that $e - xy$ is invertible in $\mathcal{A}$ if $e - yx$ is invertible.

(ii) If $\lambda \in \mathbb{C} \setminus \{0\}$ and $\lambda \in \sigma(xy)$, prove that $\lambda \in \sigma(yx)$. Thus $\sigma(xy) \cup \{0\} = \sigma(yx) \cup \{0\}$.

(iii) Show, however, that in general $\sigma(xy)$ is not always equal to $\sigma(yx)$.

3. Prove that there are no $x, y \in \mathcal{A}$ satisfying $xy - yx = e$.

4. Let $A = \mathcal{M}(\mathbb{R})$ be the Banach algebra of complex finite regular Borel measures on $\mathbb{R}$, equipped with the total variation norm and convolution as multiplication (the identity element is δ_0). For $\mu \in \mathcal{M}(\mathbb{R})$, define its Fourier–Stieltjes transform by

$$\widehat{\mu}(t) = \int_{\mathbb{R}} e^{-itx} d\mu(x), \quad t \in \mathbb{R}.$$

(i) Show that for every $t \in \mathbb{R}$, the map $\varphi_t(\mu) = \widehat{\mu}(t)$ is a character of the Banach algebra $\mathcal{M}(\mathbb{R})$.

(ii) Prove that $\{\widehat{\mu}(t) : t \in \mathbb{R}\} \subseteq \sigma(\mu)$.

(iii) Compute the spectrum of δ_a , $a \in \mathbb{R}$.

(iv) Let μ be an invertible positive measure. Show that its inverse is not positive, unless μ is concentrated at a single point.

5. (i) Let $\mathbb{T}$ denote the unit circle on the complex plane. Let $\mathcal{A}$ be the closed subalgebra of $C(\mathbb{T})$, generated by 1 and z . Prove that z is not invertible in $\mathcal{A}$. Compute the spectral radius of z (in $\mathcal{A}$).

(ii) Show that if $\mathcal{A}$ is a closed subalgebra of $\mathcal{B}$ and $x \in \mathcal{A}$, then $\sigma_{\mathcal{B}}(x) \subseteq \sigma_{\mathcal{A}}(x)$ and in general equality does not hold.

(iii) Under the notation of (ii), prove that if V is the unbounded connected component of $\mathbb{C} \setminus \sigma_{\mathcal{B}}(x)$, then $\sigma_{\mathcal{A}} \subset \mathbb{C} \setminus V$.

6. Let $x \in \mathcal{A}$ be such that $\sigma(x) = K_1 \cup K_2$, where K_1, K_2 are disjoint compact subsets of $\mathbb{C}$. Assume further that Γ is a positively oriented contour which surrounds K_1 but does not surround K_2 . Define

$$P = \frac{1}{2\pi i} \int_{\Gamma} (\lambda e - x)^{-1} d\lambda.$$

Prove that $P^2 = P$, $Px = xP$ and determine the spectra of x restricted to the subspaces $P\mathcal{A}$ and $(e - P)\mathcal{A}$.

7. Under the assumptions of the previous exercise, prove that the spectra of $x_1 = Px$, $x_2 = (e - P)x$ satisfy $\sigma(x_1) \subseteq K_1 \cup \{0\}$ and $\sigma(x_2) \subseteq K_2 \cup \{0\}$.

8. Let $x \in A$, let Ω be an open set in $\mathbb{C}$, and $\sigma(x) \subset \Omega$. Prove that there exists $\delta > 0$ such that $\sigma(x + y) \subset \Omega$ for every $y \in A$ satisfying $\|y\| < \delta$.

9. Let $x \in \mathcal{A}$ and let f be holomorphic in a neighbourhood of $\sigma(x)$. Prove that if $f(\lambda)^2 = f(\lambda)$ for every $\lambda \in \sigma(x)$, then $f(x)^2 = f(x)$.

10. Let X be a Banach space and let $\Omega \subset \mathbb{C}$ be an open set. Show that a continuous function $f : \Omega \rightarrow X$ is analytic if and only if for any $z \in \Omega$ the limit $\lim_{h \rightarrow 0} (f(z + h) - f(z))/h$ exists and is finite.

Spectral theorem for compact operators

1. Compact operators and Fredholm operators

Definition 2.1. Let X and Y be Banach spaces and let $T: X \rightarrow Y$ be a linear operator. The operator T is called *compact* if it maps bounded subsets of X into relatively compact subsets of Y (i.e., for any bounded set $B \subseteq X$, the closure of $T(B)$ in Y is compact). Let $\mathcal{K}(X, Y) = \{\text{the class of all compact } T: X \rightarrow Y\}$.

We will use the following well-known properties of compact operators.

Remark 2.2. (i) Any compact operator is bounded.

(ii) The operator $T: X \rightarrow Y$ is compact if and only if $T(B_X)$, the image of the unit ball in X , is relatively compact.

(iii) The operator $T: X \rightarrow Y$ is compact if and only if for any bounded sequence $(x_n)_{n \geq 1} \subset X$, the image $(Tx_n)_{n \geq 1} \subset Y$ contains a convergent subsequence.

(iv) The operator $T: X \rightarrow Y$ is compact if $T(B_X)$ is totally bounded in Y (i.e., for any $\varepsilon > 0$, $T(B_X)$ can be covered by a finite number of balls $B(y_i, \varepsilon)$, with $y_i \in Y$).

(v) $\mathcal{K}(X, Y)$ is a closed subspace of $\mathcal{B}(X, Y)$.

(vi) If $T \in \mathcal{K}(X, Y)$ and $S \in \mathcal{B}(Y, Z)$, $U \in \mathcal{B}(V, X)$, then $TU: V \rightarrow Y$ and $ST: X \rightarrow Z$ are compact. In particular, compact operators form a two-sided ideal in $\mathcal{B}(X)$.

(vii) An operator $T: X \rightarrow Y$ is compact if and only if $T^*: Y^* \rightarrow X^*$ is compact.

Example 2.3. Let K be a compact metric space and let μ be a finite Borel measure on K . For a fixed continuous $k: K \times K \rightarrow \mathbb{C}$, define the integral operator $T: C(K) \rightarrow C(K)$ by

$$Tf(x) = \int_K k(x, y)f(y)d\mu(y), \quad x \in K.$$

Then T is compact.

Proof. Since k is continuous on the compact space $K \times K$, it is bounded and uniformly continuous. Let $M = \|k\|_\infty < \infty$. For $f \in C(K)$ with $\|f\|_\infty \leq 1$, we have

$$|Tf(x)| \leq \int_K |k(x, y)||f(y)|d\mu(y) \leq M\mu(K), \quad x \in K,$$

and hence $T(B_{C(K)})$ is bounded. Moreover, for $x, x' \in K$ and $f \in B_{C(K)}$,

$$|Tf(x) - Tf(x')| \leq \int_K |k(x, y) - k(x', y)|d\mu(y) \leq \mu(K) \sup_{y \in K} |k(x, y) - k(x', y)|.$$

Since k is uniformly continuous on $K \times K$, the above inequality implies that the image $T(B_{C(K)})$ is equicontinuous. Thus, by the Arzelà–Ascoli theorem, we deduce the compactness of $T(B_{C(K)})$ in $C(K)$. $\square$

Example 2.4. Suppose that $L^2(X) = L^2(X, \mu)$ and let $k = k(x, y) \in L^2(X \times X, \mu \otimes \mu)$. Define the operator $T: L^2(X) \rightarrow L^2(X)$ by

$$Tf(x) = \int_X k(x, y)f(y)d\mu(y).$$

Then T is compact.

Proof. By the Cauchy–Schwarz inequality we have

$$|Tf(x)| \leq \left(\int_X |k(x, y)|^2 d\mu(y) \right)^{1/2} \left(\int_X |f(y)|^2 d\mu(y) \right)^{1/2} = \|f\|_2 \left(\int_X |k(x, y)|^2 d\mu(y) \right)^{1/2}.$$

This implies $\|Tf\|_2^2 \leq \|f\|_2^2 \|k\|_2^2 < \infty$ and hence T is bounded: $\|T\| \leq \|k\|_2$. To show the compactness, let $(g_\alpha)_\alpha$ be an orthonormal basis of $L^2(X)$. Then $(g_\alpha(x)g_\beta(y))_{\alpha, \beta}$ is an orthonormal basis of $L^2(X \times X)$, so

$$k(x, y) = \sum_{\alpha, \beta} a_{\alpha, \beta} g_\alpha(x) g_\beta(y),$$

where the coefficients $a_{\alpha, \beta}$ satisfy $\sum_{\alpha, \beta} |a_{\alpha, \beta}|^2 < \infty$. In particular, there are at most countably many nonzero terms under the sum, so after relabeling the nonzero coefficients we may write

$$k(x, y) = \sum_{i, j=1}^{\infty} a_{ij} g_i(x) g_j(y).$$

Define the truncated kernel $k_n(x, y) := \sum_{i, j=1}^n a_{ij} g_i(x) g_j(y)$ and let $T_n f(x) := \int_X k_n(x, y) f(y) d\mu(y)$. Then

$$T_n f(x) = \sum_{i, j=1}^n a_{ij} g_i(x) \int_X g_j(y) f(y) d\mu(y).$$

has finite rank and is therefore compact. Moreover, by the previous calculation and Parseval's identity,

$$\|T - T_n\|^2 \leq \|k - k_n\|_2^2 = \sum_{\max\{i, j\} > n} |a_{ij}|^2 \xrightarrow{n \rightarrow \infty} 0.$$

Therefore T is a limit of compact operators, and hence it is also compact. $\square$

Definition 2.5. Let X, Y be Banach spaces. Then $T : X \rightarrow Y$ is a Fredholm operator, if it is bounded, $\dim \text{Ker } T < \infty$ and $\text{codim } T(X) < \infty$. The space of all Fredholm operators from X to Y is denoted by $\text{Fred}(X, Y)$.

As is readily seen, every isomorphism $T : X \rightarrow Y$ is Fredholm. Roughly speaking, a general Fredholm operator behaves much like an isomorphism, except for finite-dimensional defects.

Proposition 2.6. Let $T : X \rightarrow Y$ be a bounded operator with $\text{codim } T(X) < \infty$. Then $T(X)$ is closed.

Proof. Let $X_0 = X / \text{Ker } T$ and let $\tilde{T} : X_0 \rightarrow Y$ be the injective operator induced by T ; in particular, we have $\tilde{T}(X) = T(X)$. Since the codimension of $T(X)$ is finite, there exist $y_1, y_2, \dots, y_n \in Y$ such that each y can be written uniquely in the form $y = y_0 + \alpha_1 y_1 + \alpha_2 y_2 + \dots + \alpha_n y_n$, where $\alpha_i \in \mathbb{C}$ and $y_0 \in T(X)$. Consider the auxiliary operator

$$S : X_0 \oplus \mathbb{R}^n \rightarrow Y, \quad S(x, (\alpha_1, \alpha_2, \dots, \alpha_n)) = \tilde{T}x + \sum_{k=1}^n \alpha_k y_k.$$

This operator is obviously bijective, so its inverse is bounded by the Open Mapping Theorem, and thus S is an isomorphism. Hence $T(X) = S(X_0 \oplus \{0\})$ is a closed set. $\square$

The next result provides a rich source of examples of Fredholm operators.

Theorem 2.7. If the operator $T : X \rightarrow X$ is compact, then $S = I - T$ is a Fredholm operator.

Proof. (i) The kernel $\text{Ker}(I - T) = \{x \in X : Tx = x\}$ is finite-dimensional, otherwise $T|_{\text{Ker}(I-T)} = I$ would not be compact.

(ii) The image $S(X)$ is closed. To see this, note that the quotient space $X_0 = X / \text{Ker } S$ is closed and the induced map $\tilde{S} : X_0 \rightarrow X$ is injective. If $S(X) = \tilde{S}(X_0)$ were not closed, then $\tilde{S}$ would not be an isomorphism onto its image. Then there is a sequence $(\tilde{x}_n)_{n \geq 1} \subset X_0$ with $\|\tilde{x}_n\| = 1$ for all n and $\|\tilde{S}\tilde{x}_n\| \xrightarrow{n \rightarrow \infty} 0$ (indeed, otherwise $\tilde{S}^{-1} : S(X) \rightarrow X_0$ would be bounded and hence $\tilde{S}$ would be an isomorphism). By the definition of a quotient space, this implies the existence of a sequence $(x_n)_{n \geq 1}$ such that $[x_n] = x_n + \text{Ker } S = \tilde{x}_n$, $1 \leq \|x_n\| \leq 2$ and $Sx_n = \tilde{S}\tilde{x}_n \rightarrow 0$. The latter convergence is equivalent to $(I - T)x_n = x_n - Tx_n \rightarrow 0$.

Using the compactness of T , we may pick a subsequence $(x_{n_k})_{k \geq 1}$ such that Tx_{n_k} converges to some $\bar{x}$. By the previous observation, we obtain $x_{n_k} \rightarrow \bar{x}$; but S is bounded, so we get $S(\bar{x}) = \lim_{n \rightarrow \infty} S(x_{n_k}) = 0$ and hence $\bar{x} \in \text{Ker } S$. This gives a contradiction: we have $\tilde{x}_{n_k} \rightarrow [\bar{x}] = 0$ and $\|\tilde{x}_{n_k}\| = 1$.

(iii) The image $S(X)$ has a finite codimension. Indeed, otherwise we have $S(X) = H_0 \subset H_1 \subset H_2 \subset \dots$, where $(H_n)_{n \geq 1}$ is a sequence of closed subspaces of X with $\dim(H_{n+1}/H_n) = 1$. By Riesz' lemma, for each $n \geq 1$ we can find $h_n \in H_n$ satisfying $\|h_n\| = 1$ and $\text{dist}(h_n, H_{n-1}) > 1/2$. Consequently, for any $n > m$ we may write

$$\|Th_n - Th_m\| = \|h_n - Sh_n - h_m + Sh_m\| > 1/2,$$

since $-Sh_n - h_m + Sh_m \in H_{n-1}$. Therefore, there is no subsequence of $(h_n)_{n \geq 1}$ satisfying the Cauchy condition; thus T cannot be a compact operator. $\square$

Definition 2.8. We define the index of a Fredholm operator $S : X \rightarrow Y$ by $\text{ind } S = \dim \text{Ker } S - \text{codim } S(X)$.

Theorem 2.9. Let X, Y be Banach spaces. Then the set $\text{Fred}(X, Y)$ is open in $\mathcal{B}(X, Y)$ and the index $\text{ind} : \text{Fred}(X, Y) \rightarrow \mathbb{Z}$ is continuous.

Proof. (i) The set $\{T : X \rightarrow Y \text{ is an isomorphism}\}$ is open in $\mathcal{B}(X, Y)$. Indeed, if T is an isomorphism and $S : X \rightarrow Y$ is a bounded operator such that $\|T - S\| \leq \|T^{-1}\|^{-1}$, then $\|I - T^{-1}S\| = \|T^{-1}(T - S)\| < 1$ and hence $T^{-1}S$ is invertible; it remains to note that $S[(T^{-1}S)^{-1}T^{-1}] = [(T^{-1}S)^{-1}T^{-1}]S = I$, so S is invertible.

(ii) The set $\text{Fred}(X, Y)$ is open in $\mathcal{B}(X, Y)$. To see this, assume that $S \in \text{Fred}(X, Y)$ and let $N = \text{Ker } S$. Since N is finite-dimensional, it is complemented in X : there is a closed subspace $G \subset X$ such that $X = N \oplus G$. Furthermore, by the proof of Proposition 2.6, we have $Y = S(X) \oplus H$ for some finite dimensional subspace $H \subset Y$. Consider the map $\varphi_S : G \oplus H \rightarrow Y$ by $\varphi_S(g, h) = S(g) + h$. Since $S|_G$ is injective and $S(G) = S(X)$, φ_S is an isomorphism; let us record that $\text{codim } S(G) = \dim H$. Let $T : X \rightarrow Y$ be a bounded operator and let $\varphi_T : G \oplus H \rightarrow Y$ be given as previously, by $\varphi_T(g, h) = T(g) + h$. Then if T is sufficiently close to S , then φ_T is close to φ_S and hence it is an isomorphism. This in particular means that $T|_G$ is injective, so $\dim \text{Ker } T \leq \dim N = \dim \text{Ker } S < \infty$. Furthermore, since $\varphi_T(G \oplus H) = Y$ and H is finite-dimensional, we get that $T(G) = \varphi_T(G \oplus \{0\})$ has finite codimension (equal to $\dim H$), and hence $T(X)$ also has this property. Thus T is a Fredholm operator.

(iii) The index is continuous. Under the notation in (ii), we have proved that $S|_G : G \rightarrow S(X)$ is an isomorphism and hence there is $c > 0$ such that $\|Sg\| \geq c\|g\|$ for all $g \in G$. Now, if $T \in \text{Fred}(X, Y)$ satisfies $\|T - S\| < c/2$, then $\|Tg\| \geq \frac{c}{2}\|g\|$ and hence $T|_G$ is injective. The subspace $\text{Ker } T \oplus G$ has finite codimension, so it is complemented (see the proof of Proposition 2.6): there is a finite-dimensional subspace M such that $X = \text{Ker } T \oplus G \oplus M$ and $T|_{G \oplus M}$ is an isomorphism onto $T(X) = T(G) \oplus T(M)$. However, we have proved in (ii) that $\text{codim } S(G) = \dim H = \text{codim } S(X)$; this yields

$$\text{ind } S = \dim \text{Ker } S - \text{codim } S(X) = \dim \text{Ker } S - \text{codim } S(G)$$

and $\text{ind } T = \dim \text{Ker } T - \text{codim } T(X) = (\dim \text{Ker } S - \dim M) - (\text{codim } T(G) - \dim M) = \text{ind } S$. $\square$

Corollary 2.10. (i) For a compact operator $T : X \rightarrow X$ we have $\text{ind}(I - T) = 0$.

(ii) For a compact operator $T : X \rightarrow X$, the operator $I - T$ is injective if and only if it is invertible.

Proof. (i) The mapping $\lambda \mapsto I - \lambda T$ is continuous from $[0, 1]$ to $\text{Fred}(X, X)$ and satisfies $\text{ind } I = 0$.

(ii) By the definition of the index, $I - T$ is injective if and only if $\text{codim}(I - T) = 0$, so $I - T$ is surjective. Thus the claim follows from the Open Mapping Theorem. $\square$

2. Spectral theorem for compact operators

Definition 2.11. A complex number α is an eigenvalue of a linear operator $T : X \rightarrow X$, if there exists a nonzero vector $x \in X$ such that $Tx = \alpha x$.

Lemma 2.12. If $T : X \rightarrow X$ is compact and $\alpha \in \sigma(T) \setminus \{0\}$, then α is an eigenvalue of T .

Proof. If $\alpha \in \sigma(T) \setminus \{0\}$, then $\alpha I - T$, and hence also $I - \alpha^{-1}T$, is not invertible. By Corollary 2.10, the operator $I - \alpha^{-1}T$ is not injective; this is precisely the claim. $\square$

Lemma 2.13. If $T : X \rightarrow X$ is compact and X is infinite-dimensional, then $0 \in \sigma(T)$.

Proof. If $0 \notin \sigma(T)$, then T would be invertible and hence, by Remark 2.2 (vi), $I = T^{-1}T$ would be compact. But the latter fact holds true for finite-dimensional X only. $\square$

One easily finds examples showing that in general, 0 need not be an eigenvalue of a compact operator.

Proposition 2.14. *Let $T : X \rightarrow X$ be a compact operator and $\alpha \in \sigma(T) \setminus \{0\}$. Then there exists a positive integer r such that $\text{Ker}(T - \alpha I)^r = \text{Ker}(T - \alpha I)^n$ for $n \geq r$. Furthermore, the space $\text{Ker}(T - \alpha I)^r$ is finite-dimensional.*

Proof. (i) Observe that for any operator $S : X \rightarrow X$ the kernel $\text{Ker } S^n$ either strictly increases, or stabilizes after a finite number of steps. Indeed, suppose that $\text{Ker } S^{n+1} = \text{Ker } S^n$ and $\text{Ker } S^{n+1} \subsetneq \text{Ker } S^{n+2}$. Then there exists $x \in \text{Ker } S^{n+2} \setminus \text{Ker } S^{n+1}$. The requirement $x \notin \text{Ker } S^{n+1}$ implies that $x \notin \text{Ker } S^n$ and hence the nonzero vector $z = Sx$ must belong to $\text{Ker } S^{n+1} = \text{Ker } S^n$. Thus $S^n(Sx) = S^{n+1}x = 0$, a contradiction.

(ii) Suppose that the assertion does not hold: by the previous step, $\text{Ker}(T - \alpha I) \subsetneq \text{Ker}(T - \alpha I)^2 \subsetneq \dots$. By Riesz' lemma, there exists a sequence $(x_n)_{n \geq 1} \subset X$ such that $\|x_n\| = 1$, $x_n \in \text{Ker}(T - \alpha I)^n$ and $\text{dist}(x, \text{Ker}(T - \alpha I)^{n-1}) > 1/2$. Then for $m < n$ we have

$$\|Tx_n - Tx_m\| = \left\| \alpha x_n + \left((T - \alpha I)x_n - \alpha x_m - (T - \alpha I)x_m \right) \right\|$$

and the vector in the parentheses lies in $\text{Ker}(T - \alpha I)^{n-1}$, since so do $(T - \alpha I)x_n$, αx_m and $(T - \alpha I)x_m$. This yields $\|Tx_n - Tx_m\| \geq \frac{1}{2}|\alpha|$ and hence $(Tx_n)_{n \geq 1}$ does not contain a convergent subsequence. This contradicts the compactness of T .

(iii) Finally, to show that $\dim \text{Ker}(T - \alpha I)^r < \infty$, simply note that $(T - \alpha I)^r$ is a Fredholm operator:

$$(T - \alpha I)^n = (-\alpha)^n I + \sum_{k=1}^n \binom{n}{k} (-\alpha)^{n-k} T^k = (-\alpha)^n I + \text{a compact operator.} \quad \square$$

Definition 2.15. Under the notation of the above proposition, the smallest admissible value of $r = r(\alpha)$ is called the multiplicity of α .

Definition 2.16. Let $S : X \rightarrow X$ be a linear operator. A subspace $Y \subseteq X$ is invariant with respect to S , if $S(Y) \subseteq Y$.

Theorem 2.17. *Let $T : X \rightarrow X$ be a compact operator and let α be a nonzero eigenvalue of T , with multiplicity $r = r(\alpha)$. Then we have the decomposition*

$$X = \text{Ker}(T - \alpha I)^r \oplus (T - \alpha I)^r(X).$$

Each of the subspaces $\text{Ker}(T - \alpha I)^r$, $(T - \alpha I)^r(X)$ is invariant with respect to T . If $\beta \neq \alpha$ is another nonzero eigenvalue of T , or multiplicity s , then $\text{Ker}(T - \beta I)^s \subseteq (T - \alpha I)^r(X)$.

Proof. To show that $\text{Ker}(T - \alpha I)^r$ is invariant, pick a vector x from this subspace and note that $(T - \alpha I)^r T x = T(T - \alpha I)^r x = 0$. One checks similarly that $(T - \alpha I)^r(X)$ is invariant. Next, in order to establish the decomposition, it is enough to prove that $\text{Ker}(T - \alpha I)^r \cap (T - \alpha I)^r(X) = \{0\}$. To this end, pick a vector x from the intersection: we have $(T - \alpha I)^r x = 0$ and $x = (T - \alpha I)^r y$ for some $y \in X$, and hence $(T - \alpha I)^{2r} y = 0$. Thus, by the very definition of the multiplicity, we have $(T - \alpha I)^r y = 0$ and therefore $x = 0$.

Finally, pick a nonzero eigenvalue $\beta \neq \alpha$ and set $S = (T - \alpha I)^r$, $Q = (S - \beta I)^s$. The operators Q , S commute, which implies $Q(\text{Ker } S) \subseteq \text{Ker } S$, $Q(S(X)) \subseteq S(X)$. Therefore, if we fix $x \in \text{Ker } Q$ and write down the unique decomposition $x = y + z$ for $y \in \text{Ker } S$ and $z \in S(X)$, then we have $0 = Qx = Qy + Qz$ and $Qy \in \text{Ker } S$, $Qz \in S(X)$. Since $\text{Ker } S \cap S(X) = \{0\}$, as we have shown above, this gives $Qy = 0$. Recall that we also have $Sy = 0$; now we will prove that $y = 0$, which will give the claim. To this end, we write

$$I = a(T - \alpha I) + b(T - \beta I) \quad \text{for } a = \frac{1}{\beta - \alpha}, b = \frac{1}{\alpha - \beta}.$$

If we pick a sufficiently large integer N , then

$$I = I^N = [a(T - \alpha I) + b(T - \beta I)]^N = \sum_{k=0}^N \binom{N}{k} a^k b^{N-k} (T - \alpha I)^k (T - \beta I)^{N-k}$$

can be rewritten as $AQ + BS$ for some operators A and B . This gives $y = AQy + BSy = 0$, as desired. $\square$

We summarize the above discussion in the following.

Theorem 2.18. *Let X be a Banach space and let $T: X \rightarrow X$ be a compact operator. Then*

- (i) $0 \in \sigma(T)$ whenever X is infinite-dimensional.
- (ii) Every nonzero point of the spectrum is an eigenvalue.
- (iii) Every nonzero eigenvalue has finite algebraic multiplicity: $\dim \ker(T - \lambda I) < \infty$ for $\lambda \neq 0$.
- (iv) The set of nonzero eigenvalues is at most countable, and its only possible accumulation point is 0.
- (v) If $\lambda_1, \lambda_2, \dots, \lambda_n$ are distinct nonzero eigenvalues, then the corresponding generalized eigenspaces $E_{\lambda_j} = \ker(T - \lambda_j I)^{r(\lambda_j)}$ are finite-dimensional and linearly independent.

Proof. We only need to establish (iv): for any $\varepsilon > 0$ there is a finite number of eigenvalues α of T with $|\alpha| > \varepsilon$. Indeed, if $\alpha_1, \alpha_2, \dots$ are distinct eigenvalues of T with $|\alpha_j| > \varepsilon$, we let $w_1, w_2, \dots$ be the corresponding normalized eigenvectors. By the previous theorem, $\{w_j\}_{j \geq 1}$ are linearly independent. Let $H_0 = \{0\}$ and $H_n = \text{lin}(w_1, w_2, \dots, w_n)$ for $n \geq 1$. Then $H_0 \subset H_1 \subset H_2 \subset \dots$ and by Riesz' lemma, we can find a sequence $(x_n)_{n \geq 1}$ of vectors satisfying $x_n \in H_n$, $\|x_n\| = 1$ and $\text{dist}(x_n, H_{n-1}) > 1/2$. If we write the unique decomposition $x_n = \beta_1 w_1 + \beta_2 w_2 + \dots + \beta_n w_n$, then $\beta_1 w_1 + \beta_2 w_2 + \dots + \beta_{n-1} w_{n-1} \in H_{n-1}$ and therefore

$$\|\beta_n w_n\| = \|x_n - (\beta_1 w_1 + \beta_2 w_2 + \dots + \beta_{n-1} w_{n-1})\| > 1/2,$$

so that $|\beta_n| > 1/2$. It remains to note that for $m < n$,

$$\|Tx_n - Tx_m\| = \left\| \sum_{k=1}^n \alpha_k \beta_k^{(n)} w_k - \sum_{k=1}^m \alpha_k \beta_k^{(m)} w_k \right\| = \left\| \alpha_n \beta_n^{(n)} w_n - v \right\| = |\alpha_n \beta_n^{(n)}| \left\| w_n - \frac{v}{\alpha_n \beta_n^{(n)}} \right\|$$

for some vector $v \in H_{n-1}$. Thus $\|Tx_n - Tx_m\| > \varepsilon/2$, which implies that $(Tx_n)_{n \geq 1}$ does not have a convergent subsequence, thus contradicting the compactness of T . $\square$

3. Exercises

1. For a fixed $f \in L_1(\mathbb{R})$ and a fixed $1 \leq p \leq \infty$, we consider the operator $T_f: L_p(\mathbb{R}) \rightarrow L_p(\mathbb{R})$ given by $T_f g = f * g$. Prove that T_f is compact.

2. Let $T: X \rightarrow Y$ be a Fredholm operator. Prove that $T^*: Y^* \rightarrow X^*$ is also a Fredholm operator and $\text{ind } T^* = -\text{ind } T$.

3. Let K be a compact set. For a fixed $f \in C(K)$, we consider the operator $T_f: C(K) \rightarrow C(K)$, given by $T_f g = f \cdot g$. Prove that T_f is a Fredholm operator if and only if $f^{-1}(0) \subset K$ is finite, open and closed.

4. Prove that the composition of two Fredholm operators S, T is again a Fredholm operator and that $\text{ind}(ST) = \text{ind}(S) + \text{ind}(T)$.

5. Let $1 \leq p \leq \infty$. Let $\mathbb{D}$ be a unit disc in $\mathbb{C}$, equipped with the Lebesgue measure and let $B_p(\mathbb{D})$ be the space of all analytic functions f on $\mathbb{D}$ such that $\|f\|_p = \left(\int_{\mathbb{D}} |f(z)|^p dz \right)^{1/p} < \infty$ (with obvious modification for $p = \infty$).

(i) Prove that $B_p(\mathbb{D})$ is a Banach space.

(ii) For a bounded analytic function $f: \mathbb{D} \rightarrow \mathbb{C}$, we consider the operator $T_f: B_p(\mathbb{D}) \rightarrow B_p(\mathbb{D})$ given by $T_f g = f \cdot g$. Prove that if T_f is compact, then $T_f = 0$.

(iii) Prove that T_f is a Fredholm operator if and only if there are $\delta > 0$ and $0 < r < 1$ such that $|f(z)| > \delta$ for all $z \in \mathbb{D}$ with $|z| > r$.

6. A bounded linear operator $T: X \rightarrow X$ has the property that $T^n: X \rightarrow X$ is compact for some $n = 1, 2, \dots$

(i) Does this imply that T is compact?

(ii) Prove that $T - \lambda I$, $\lambda \neq 0$, is a Fredholm operator of index 0.

(iii) Show that $\sigma(T)$ is at most countable and if $\lambda \in \sigma(T) \setminus \{0\}$, then λ is an eigenvalue of T .

7. Prove that if $T: X \rightarrow Y$ is a Fredholm operator and $S: X \rightarrow Y$ is a compact operator, then $S + T$ is Fredholm and $\text{ind}(S + T) = \text{ind } T$.

Bibliography

- [1] W. Rudin, Analiza Funkcjonalna, Wydawnictwo Naukowe PWN, 2001.