

Selected problems for Stochastic Processes: part II

1. Consider the system

$$\begin{aligned}dX_t &= Y_t dt, \\dY_t &= -\beta X_t dt - \alpha Y_t dt + \sigma dB_t,\end{aligned}$$

where B is a standard one-dimensional Brownian motion and β , σ and α are positive constants. Solve this system explicitly. Is the solution a Gaussian process?

2. Let $(B_t)_{t \geq 0}$ be a standard Brownian motion and consider the diffusion

$$X_t = x + \mu t + \sigma B_t,$$

where $\mu \in \mathbb{R}$ and $\sigma > 0$. Using the Feynman-Kac formula, compute explicitly

$$u(t, x) = \mathbb{E} \exp \left(kX_T - \lambda \int_t^T X_s ds \right),$$

where $\lambda > 0$ and $k \in \mathbb{R}$.

3. Let $(B_t)_{t \geq 0}$ be a standard Brownian motion and consider the diffusion

$$dX_t = -\theta X_t dt + \sigma dB_t, \quad X_0 = x,$$

where $\theta > 0$ and $\sigma > 0$. Show that

$$X_t = xe^{-\theta t} + \sigma \int_0^t e^{-\theta(t-s)} dW_s.$$

Compute $\mathbb{E}[X_t]$ and $\text{Var}(X_t)$. Show that $M_t = e^{\theta t} X_t$ is not a martingale. Find a martingale involving $(X_t)_{t \geq 0}$. Determine constants a and b such that $N_t = e^{at}(X_t^2 + b)$ is a martingale.

4. Consider the stochastic differential equation

$$dX_t = \frac{1}{X_t} dt + dW_t, \quad X_0 = x > 0.$$

Prove that there exists a unique strong solution on $T = \mathbb{R}_+$. Derive the stochastic differential equation satisfied by the process $R_t = X_t^2$.

5. Consider the stochastic differential equation

$$dX_t = \sqrt{X_t} dt + dB_t, \quad X_0 = 1.$$

Prove that with probability 1, the solution reaches 0 in finite time.

6. Let $(B_t)_{t \geq 0}$ be a one-dimensional Brownian motion and let $\tau_0 = \inf\{t > 0 : B_t = 0\}$. For $x, \lambda > 0$, compute

$$u(x) = \mathbb{E}_x \exp \left(-\lambda \int_0^{\tau_0} \frac{1}{B_s^2} ds \right).$$

7. Consider the domain $D = \{(x, y) \in \mathbb{R}^2 : |y| < x\}$. Let $B = (B^{(1)}, B^{(2)})$ be a two dimensional Brownian motion and let $\tau = \inf\{t > 0 : B_t \in \partial D\}$. For $(x, y) \in D$, compute $\mathbb{P}_{x,y}(B_\tau^{(2)} > 0)$.

Hint: Maybe a function $u(x, y) = \arctg \frac{y}{x}$ will help.

8. For fixed $a, \theta > 0$, consider the diffusion

$$dX_t = (a - \theta X_t) dt + dB_t, \quad X_0 = x.$$

Compute $\mathbb{E}X_t^2$.

9. Let B be a one dimensional Brownian motion and let

$$\tau = \inf \left\{ t \geq 0 : B_t^2 + \left(\int_0^t B_s ds \right)^2 = 1 \right\}.$$

Compute $\mathbb{E} \left(B_\tau^4 - 6B_\tau \int_0^\tau B_s ds \right)$.