

Selected problems for Stochastic Processes

Unless stated otherwise, B denotes the one-dimensional Brownian motion started at zero. The problems in parts I and II are formulated as yes/no questions.

I. Basics

1. Let $(X_t)_{t \geq 0}$ be a Markov process on a metric space $(E, \mathcal{E})$.

- ☐ For any Borel function $f : E \rightarrow E$, the process $(f(X_t))_{t \geq 0}$ is Markovian.
- ☐ For any one-to-one Borel function $f : E \rightarrow E$, the process $(f(X_t))_{t \geq 0}$ is Markovian.
- ☐ The process $((t^2, X_t))_{t \geq 0}$ with values in $\mathbb{R}_+ \times E$ is Markovian.

2. Let X be a homogeneous Markov process. Then

- ☐ we have $\mathbb{P}(X_{t_n} \in \Gamma_n | X_{t_{n-1}} \in \Gamma_{n-1}, \dots, X_{t_1} \in \Gamma_1) = \mathbb{P}(X_{t_n} \in \Gamma_n | X_{t_{n-1}} \in \Gamma_{n-1})$ for all $t_1 < t_2 < \dots < t_n$ and all $\Gamma_1, \Gamma_2, \dots, \Gamma_n$ such that $\mathbb{P}(X_{t_{n-1}} \in \Gamma_{n-1}, \dots, X_{t_1} \in \Gamma_1) > 0$.
- ☐ for any $u \geq 0$, the process $(X_{t+u})_{t \geq 0}$ is a homogeneous Markov process with the same transition function as X .
- ☐ if the family induced by X is strongly Markovian, then for any stopping time τ of X the process $(X_{t+\tau})_{t \geq 0}$ is a homogeneous Markov process with the same transition function as X .

3. Consider the following time-transformations of B .

- ☐ The process $(B_{2t})_{t \geq 0}$ is a homogeneous Markov process.
- ☐ The process $(B_{t^2})_{t \geq 0}$ is a homogeneous Markov process.
- ☐ The process $(B_{2t} + B_{t^2})_{t \geq 0}$ is a Markov process.

4. The probability $\mathbb{P}(B_1 > 1, B_2 > 2, B_3 > 3)$ is equal to

- ☐ $\mathbb{P}(B_2 > 2, B_4 > 4, B_6 > 6)$.
- ☐ $\mathbb{P}(B_2 > 1 + B_1, B_3 > 1 + B_1, B_4 > 1 + B_1)$.
- ☐ $\mathbb{P}(B_1 > 1, B_{1/2} > 1, B_{1/3} > 1)$.

5. Let M, N be two independent Poisson processes with intensities μ and λ , respectively.

- ☐ The process $X_t = \max\{M_t, N_t\}$, $t \geq 0$, is a Markov process.
- ☐ The process $Y_t = M_t + N_t$, $t \geq 0$, is a Poisson process.
- ☐ The process $Z_t = M_t N_t$, $t \geq 0$, is a Poisson process.

6. Let X be a homogeneous, continuous Markov chain on $\{0, 1, 2\}$, with the transition matrix

$$P(t) = \begin{pmatrix} \frac{1}{3} + \frac{2}{3}e^{-3t} & \frac{1}{3} - \frac{1}{3}e^{-3t} & \frac{1}{3} - \frac{1}{3}e^{-3t} \\ \frac{1}{3} - \frac{1}{3}e^{-3t} & \frac{1}{3} + \frac{2}{3}e^{-3t} & \frac{1}{3} - \frac{1}{3}e^{-3t} \\ \frac{1}{3} - \frac{1}{3}e^{-3t} & \frac{1}{3} - \frac{1}{3}e^{-3t} & \frac{1}{3} + \frac{2}{3}e^{-3t} \end{pmatrix}.$$

- ☐ The transition matrix of an embedded Markov chain is $\Gamma(t) = \begin{pmatrix} 0 & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{3} & 0 & \frac{2}{3} \\ \frac{1}{2} & \frac{1}{2} & 0 \end{pmatrix}$.
- ☐ The expected number of transitions between two successive visits to 1 is equal to 4.

☐ If X starts from 0, then $\lim_{t \rightarrow \infty} \frac{1}{t} \int_0^t X_s^2 ds = \frac{19}{12}$ almost surely.

7. Let τ be an exponential random variable. We let

$$X_t = \begin{cases} X_0 + t & \text{if } X_0 > 0 \text{ or } X_0 + t \leq 0, \\ 0 & \text{if } X_0 < 0 \text{ and } 0 < t + X_0 < \tau, \\ X_0 + t - \tau & \text{if } X_0 < 0 \text{ and } X_0 + t > \tau. \end{cases}$$

The process X is therefore a uniform motion to the right with a holding time of length τ at zero.

☐ X is a Markov process.

☐ The family induced by X is strongly Markovian.

☐ The generator A of X given by $Af = \begin{cases} f'(x) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$ for a suitably regular function f .

8. Let $Q = (q_{ij})_{i,j \geq 0}$ be a q -matrix of a regular jump Markov chain X on $\mathbb{Z}_+$.

☐ If Q has bounded entries, then Q is non-explosive.

☐ If Q has unbounded entries, then Q is explosive.

☐ If Q is a bounded operator on ℓ_∞ and Q is a q -matrix of another continuous Markov chain Y on $\mathbb{Z}_+$, then X and Y have the same law.

9. Let $\tau = \inf\{t \geq 0 : B_t = 1\}$, $\sigma = \inf\{t \geq \tau : B_t = 0\}$, $\eta = \inf\{t \geq \sigma : B_t = 1\}$. Then

☐ $\mathbb{P}(\eta < 3) = 2\mathbb{P}(B_3 > 3)$.

☐ $\mathbb{P}(\eta < 3) = 2\mathbb{P}(3B_1 > 1)$.

☐ $\mathbb{P}(\eta < 3) = \mathbb{P}(B_3 < -1, \sup_{t \in [0,3]} B_t > 1)$.

10. Let X be a continuous Markov chain and let Y be the embedded chain.

☐ It may happen that a stationary distribution exists for X , but not for Y .

☐ It may happen that a stationary distribution exists for Y , but not for X .

☐ It may happen that a stationary distributions for both X and Y do not exist.

II. More difficult test problems

1. Let M, N be two independent Poisson processes with intensities μ and λ , respectively.

☐ The process $X_t = M_t - N_t$, $t \geq 0$, is a Markov process.

☐ The process $X_t = M_t - N_t$, $t \geq 0$, is a compound Poisson process.

☐ The generator A of process X acts via $Af(n) = \lambda f(n+1) - (\lambda + \mu)f(n) + \mu f(n-1)$ for any $f \in B(\mathbb{Z})$ and $n \in \mathbb{Z}$.

2. Let B be a one-dimensional Brownian motion started at zero.

☐ For any $t \geq 0$, the variables $\int_0^t B_s^2 ds$ and $t^2 \int_0^1 B_s^2 ds$ have the same law.

☐ The processes $\left(\int_0^t B_s^2 ds\right)_{t \geq 0}$ and $\left(t^2 \int_0^1 B_s^2 ds\right)_{t \geq 0}$ have the same law.

☐ The process $\left(\int_0^t B_s^2 ds\right)_{t \geq 0}$ is Gaussian.

3. Let B be a one-dimensional Brownian motion started at zero.

- ☐ The integral $\int_0^t \frac{B_s}{s} ds$ is well-defined almost surely.
- ☐ The process $\beta_t = B_t - \int_0^t \frac{B_s}{s} ds$, $t \geq 0$, is a Brownian motion.
- ☐ We have $\int_0^\infty |B_s| ds = +\infty$ almost surely.

4. Let B be a one-dimensional Brownian motion and let $X_t = B_t - tB_1$, $t \in [0, 1]$, be the Brownian bridge.

- ☐ The process X is independent of B_1 .
- ☐ The process $Y_t = X_{1-t}$, $t \in [0, 1]$, is also a Brownian bridge.
- ☐ The process $Z_t = (1+t)X_{t/(t+1)}$, $t \geq 0$, is a Brownian motion.

5. For $a \in \mathbb{R}$, let $\tau_a := \inf\{t \geq 0 : B_t = a\}$.

- ☐ We have $\tau_a \sim a^2 \tau_1$.
- ☐ For $a, b > 0$ we have $\tau_{a+b} \sim \tau_a + \tilde{\tau}_b$, where $\tilde{\tau}_b$ is an independent copy of τ_b .
- ☐ $s\tau_1 + t\tilde{\tau}_1 \sim (\sqrt{s} + \sqrt{t})^2 \tau_1$ for $s, t > 0$.

6. Let $B(\mathbb{R})$ be the space of bounded functions on $\mathbb{R}$.

- ☐ The operator $Af(x) = f(x+1)$, $f \in B(\mathbb{R})$, is a generator of some Markov family on $\mathbb{R}$.
- ☐ The operator $Af(x) = x^2 f'(x)$, acting on $D(A) \subset C_c^1(\mathbb{R})$, is a generator of some Markov family on $\mathbb{R}$.
- ☐ The operator $Af(x) = \int_x^{x+1} (f(y) - f(x)) dy$, $f \in B(\mathbb{R})$, is a generator of some Markov family on $\mathbb{R}$.

7. Consider the process $X_t = B_t^2$, $t \geq 0$.

- ☐ X is a Markov process.
- ☐ The generator A of X acts by $Af(x) = 2f'(x) + 4xf''(x)$ if $f \in C_c^2(\mathbb{R}_+)$.
- ☐ $D(A)$ contains a function f satisfying $f''(0+) = +\infty$.

8. Consider the homogeneous continuous Markov family X on $E = \{0, 1, 2\}$, with the kernel

$$P(t) = \frac{1}{5} \begin{pmatrix} 2 + 3e^{-5t} & 1 - e^{-5t} & 2 - 2e^{-5t} \\ 2 - 2e^{-5t} & 1 + 4e^{-5t} & 2 - 2e^{-5t} \\ 2 - 2e^{-5t} & 1 - e^{-5t} & 2 + 3e^{-5t} \end{pmatrix}.$$

- ☐ Under $\mathbb{P}_0$, X_t is a martingale.
- ☐ There is a constant a such that $(X_t - at)_{t \geq 0}$ is a $\mathbb{P}_0$ -martingale.
- ☐ We have $\mathbb{E}_0 \int_0^\tau X_s ds = \frac{11}{7}$.

III. Regular problems

1. Do there exist functions $a(t)$, $b(t)$ such that

$$P_t = \frac{1}{7} \begin{pmatrix} a(t) & 4 - 4e^{-2t} \\ 3 - 3e^{-2t} & b(t) \end{pmatrix}$$

is a transition matrix of some time-homogeneous Markov process on a two-element state space?

2. Show that the random variables $\sup_{t \geq 0} (|B_t| - t)$ and $\left(\sup_{t \geq 0} \frac{|B_t|}{1+t} \right)^2$ have the same distribution.
3. Prove that $\frac{\log(1+B_t^2)}{\log t}$ converges in probability to 1 as $t \rightarrow \infty$.
4. (i) Prove that for all $t > 0$ and all $\varepsilon > 0$ we have $\mathbb{P} \left(\sup_{s \in [0, t]} |B_s| \leq \varepsilon \right) > 0$.
- (ii) Prove that there exists $c \in (0, \infty)$ such that for all $\varepsilon \in (0, 1)$, $\mathbb{P} \left(\sup_{s \in [0, 1]} |B_s| \leq \varepsilon \right) \geq e^{-c/\varepsilon^2}$.
- (iii) Prove that for all $t > 0$ and all $x > 0$, $\mathbb{P} \left(\sup_{s \in [0, t]} |B_s| \geq x \right) > 0$.

5. If a given particle is alive at a certain time, its additional life length is a random variable which is exponentially distributed with parameter a . Upon death, it leaves k offspring with probability p_k , $k \geq 0$. As usual, particles act independently of other particles and of the history of the process. For simplicity, assume that $p_1 = 0$. Let $X(t)$ be the number in the population at time t . Find the generator matrix and write out the forward and backward systems of differential equations.

6. Consider the $M/M/s$ queue with Poisson input of rate a , where there are s servers. Service times are i.i.d. exponential random variables with parameter b . Customers, who arrive when all s servers are busy, leave; nobody waits. Model this as a birth-death process. What are the parameters of the birth-death process? Let $Q(t)$ be the number of busy servers at time t . Find

$$p_s = \lim_{t \rightarrow \infty} \mathbb{P}(Q(t) = s).$$

This represents the probability that an arriving customer finds the system blocked and is lost. Why is this also the long run percentage of customers who are lost?

7. Let B be an n -dimensional Brownian motion and let τ_R be the first moment the process B reaches the sphere $S(0, R) \subset \mathbb{R}^n$. Compute $\mathbb{E}_0 \tau_R$ and $\mathbb{P}_{x_0}(\tau_1 < \tau_4)$, where $x_0 = (2, 0, 0, 0, \dots, 0)$.

8. Consider the geometric Brownian motion $X_t = \exp(\mu t - \sigma B_t)$, $t \geq 0$, where $\mu < 0$ and $\sigma > 0$ are fixed parameters. Given $\tau = \inf\{t : X_t = 2\}$, compute $\mathbb{E}_0 \int_0^\tau X_t^2 dt$.