

Stochastic Processes, Midterm, April 28th, 2026 r.

Duration: 180 minutes.

1. Let $(B_t)_{t \geq 0}$ be a one-dimensional Wiener process and let $(N_t)_{t \geq 0}$ be an independent Poisson process with parameter 3, $B_0 = N_0 = 0$ a.s. Compute $\mathbb{E}(B_3 - B_2 + B_1 N_1)^2$ and the almost sure limit $\lim_{t \rightarrow \infty} \frac{B_{2t}}{N_t}$.

2. A service system consists of two servers, each working with exponential service rate of two customers per hour. The maximum system capacity is three customers: two being served and one waiting in queue. Customers arrive according to a Poisson process with rate three per hour; if the system is full, arriving customers are rejected. Initially, the system is empty.

a) Describe the system as a continuous-time Markov chain on $\{0, 1, 2, 3\}$ and write the backward Kolmogorov equations for p_{00} and p_{21} .

b) What is the probability that the system is empty at the moment when the second customer arrives?

3. Consider a homogeneous continuous-time Markov chain X on $E = \{0, 1, 2, \dots\}$ with q -matrix

$$\begin{pmatrix} -1 & 1 & 0 & 0 & 0 & \dots \\ 1 & -2 & 1 & 0 & 0 & \dots \\ 0 & 2 & -4 & 2 & 0 & \dots \\ 0 & 0 & 3 & -6 & 3 & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots \end{pmatrix}$$

a) Is the chain recurrent?

b) Compute the expected return time to state 0.

c) Assume $X_0 = 2$. How many times on average will X visit state 1 before hitting 0?

4. Let $(B_t)_{t \geq 0}$ be a one-dimensional Wiener process, $B_0 = 0$ almost surely.

a) Show that $X_t = e^{B_t}$, $t \geq 0$, is a homogeneous Markov process and find its transition function.

b) Let A be the generator of the Markov semigroup associated with X . Compute Af for $f \in C_c^2(\mathbb{R}_+)$.

5. Let $(B_t)_{t \geq 0}$ be a one-dimensional Wiener process, $B_0 = 0$ almost surely. For a given $t > 0$, define $d_t = \inf\{s \geq t : B_s = 0\}$. Find the joint density of the random vector (B_1, d_1) .

6. Let X be a strong Markov process on $E = \mathbb{R}$ with continuous sample paths. The generator A of the associated Markov semigroup satisfies

$$Af(x) = \frac{1}{2} \frac{\partial^2 f(x)}{\partial x^2} - \frac{\partial f(x)}{\partial x} \quad \text{for } x \in \mathbb{R}, f \in C_c^2(\mathbb{R}).$$

Let

$$\tau = \inf\{t : |X_t| = 1\}.$$

Compute $\mathbb{E}_0 \tau$, $\mathbb{P}_0(X_\tau = 1)$, and $\mathbb{E}_0 X_{\tau+1}$.