

1. Suppose that X is a strong solution to the differential equation

$$dX_t = (2 - X_t) dt + \sqrt{X_t} dB_t, \quad X_0 = x > 0.$$

- (a) Find the drift and diffusion coefficients of X , and compute the generator of the process.
 (b) Using Itô's formula, write the SDE for the process $Y_t = X_t^2$.
 (c) Check whether the process $M_t = e^t(X_t - 2)$, $t \geq 0$, is a martingale.
 (d) Write the backward and forward equations for the transition density of X .

2. (Dynkin's formula). Let $f \in C_c^2(\mathbb{R}^n)$ and let X be a diffusion process. Let τ be a stopping time, which is integrable with respect to $\mathbb{P}_x$ for some $x \in \mathbb{R}^n$. Prove that $\mathbb{E}_x f(X_\tau) = f(x) + \mathbb{E}_x \int_0^\tau Af(X_s) ds$.

3. For a $\varphi \in C_0^2(\mathbb{R}^n)$, write down (in terms of Brownian motion) a bounded solution u of

$$\frac{\partial u(t, x)}{\partial t} - \frac{1}{2} \Delta_x u(t, x) = 0, \quad \text{for } t > 0, x \in \mathbb{R}^n,$$

with the initial condition $u(0, x) = \varphi(x)$ (from general theory, it is known that the solution is unique).

4. Let $D = \mathbb{R} \times (0, \infty)$. Find the explicit formula for the bounded solution of the problem

$$\begin{cases} \frac{\partial^2 u}{\partial x^2} + \frac{\partial u}{\partial y} = 0 & \text{on } D, \\ u(x, 0) = \cos x. \end{cases}$$

5. Write down the Kolmogorov forward and backward equations for the one-dimensional Brownian motion with a constant drift μ , and verify that the transition density satisfies these equations.

6. Let $V : \mathbb{R} \rightarrow \mathbb{R}$ be a bounded and continuous function. Using the Feynmann-Kac formula, solve

$$u_t(t, x) = \frac{1}{2} u_{xx}(t, x) - V(x)u(t, x), \quad t > 0, x \in \mathbb{R},$$

with the initial condition $u(0, x) = f(x)$ (f is a given continuous function on $\mathbb{R}$).

7. Let $(X_t)_{t \geq 0}$ satisfy the stochastic differential equation $dX_t = -\beta X_t dt + \sigma dB_t$, where $\beta, \sigma > 0$. For $a > 0$, define $\tau_a = \inf\{t \geq 0 : X_t = a\}$. Using the Feynman-Kac formula, compute $u(x) = \mathbb{E}_x[e^{-\lambda \tau_a}]$ for $x \leq a$ and $\lambda > 0$.

8. Suppose that the stock price process S satisfies the stochastic differential equation

$$dS_t = rS_t dt + \sigma S_t dB_t \quad (r > 0 \text{ is the interest rate and } \sigma > 0 \text{ is the volatility}).$$

Use the Feynman-Kac formula to show that the solution of the partial differential equation

$$\frac{\partial u}{\partial t} + \frac{1}{2} \sigma^2 x^2 \frac{\partial^2 u}{\partial x^2} + rx \frac{\partial u}{\partial x} - ru = 0,$$

for $0 \leq t < T$, with terminal condition $u(T, x) = \Phi(x)$, is given by $u(t, x) = \mathbb{E}_x[e^{-r(T-t)} \Phi(S_T)]$.

9. Let $D \subset \mathbb{R}^d$ be a given bounded open set let $g : D \rightarrow \mathbb{R}$, $f : \partial D \rightarrow \mathbb{R}$ be bounded continuous functions. Assume that $u : \overline{D} \rightarrow \mathbb{R}$ is continuous, of class $C^2(D)$, and solves the Poisson equation

$$\frac{1}{2} \Delta u = -g \quad \text{in } D,$$

subject to the boundary condition $u = f$ on ∂D . Prove that $u(x) = \mathbb{E}_x \left[f(B_{\tau_D}) + \int_0^{\tau_D} g(B_t) dt \right]$ for $x \in D$, where $\tau_D = \inf\{t \geq 0 : B_t \in \partial D\}$. In particular, the expected exit time from a ball equals

$$\mathbb{E}_x \tau_{B(0, r)} = \frac{r^2 - \|x\|^2}{d}, \quad x \in B(0, r).$$

10. For fixed $\alpha, r, K > 0$, consider the equation

$$dX_t = rX_t(K - X_t) dt + \alpha X_t(K - X_t) dB_t, \quad X_0 = x \geq 0.$$

- (a) Prove that a unique strong solution exists by proceeding as follows: for $n = 1, 2, \dots$ define

$$b_n(y) = \begin{cases} ry(K - y), & 0 \leq y \leq n, \\ rn(K - n), & y > n, \end{cases} \quad \sigma_n(y) = \begin{cases} \alpha y(K - y), & 0 \leq y \leq n, \\ \alpha n(K - n), & y > n. \end{cases}$$

Let $X^{(n)}$ be the unique solution of $dX_t^{(n)} = b_n(X_t^{(n)}) dt + \sigma_n(X_t^{(n)}) dB_t$, $X_0^{(n)} = x$. Define the stopping time $\tau_n = \inf\{t > 0 : X_t^{(n)} = n\}$ and show that $X_t^{(n)} = X_t^{(n+1)}$ for all $t < \tau_n$. Use this to find a unique strong solution X for $t < \tau_\infty := \lim_{n \rightarrow \infty} \tau_n$.

- (b) Prove that $\tau_\infty = \infty$ almost surely.
 (c) Prove that if $0 < X_0 < K$, then $0 < X_t < K$ for all t .