

Stochastic Processes — Problems for Quiz 3

In all problems below, B stands for the standard, one-dimensional Brownian motion.

1. Does there exist a time-homogeneous Markov family on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ with the generator A acting on $C_c^2(\mathbb{R})$ by a) $Af(x) = f(x+1)$; b) $Af(x) = xf'(x)$; c) $Af(x) = \int_0^\infty (f(y) - f(x))d\nu(y)$, where ν is a finite measure on $\mathbb{R}$? If the answer is in the affirmative, describe the family.

2. Let N be a Poisson process with intensity 3. Compute the generator for the process $(N_t - t)_{t \geq 0}$.

3. Solve the stochastic differential equations

a) $dX_t = X_t dt + dB_t$, $X_0 = x$.

b) $dX_t = (\alpha X_t - X_t^2)dt + \sigma X_t dB_t$, $X_0 = x > 0$. *Hint: substitute $X = 1/Z$.*

4. Show that there is a unique strong solution of the stochastic differential equation

$$dX_t = \ln(1 + X_t^2)dt + \mathbf{1}_{\{X_t > 0\}}X_t dB_t, \quad X_0 = a.$$

5. Let $(W_t)_{t \geq 0}$ be a standard d -dimensional Brownian motion, and let A be a (non-random) $d \times d$ matrix. Consider the d -dimensional stochastic differential equation

$$dX_t = AX_t dt + dW_t, \quad t \geq 0.$$

Find an explicit formula for the strong solution when the initial condition is $X_0 = x$, and show that the process X is Gaussian.

6. Let X be a diffusion with coefficients $b(x) = cx$ and $\sigma(x) = 1$. Find its generator and show that the process $X_t^2 - 2c \int_0^t X_s^2 ds - t$, $t \geq 0$, is a local martingale.

7. Consider the differential equation

$$dX_t = \mathbf{1}_{\{X_t \neq 1\}} \operatorname{sgn}(X_t) dB_t, \quad X_0 = 0.$$

Prove that there are weak solutions but there are no strong solutions. Prove that uniqueness in law does not hold.

8. Let (X, Y) be a strong solution to the (two-dimensional) differential equation

$$dX_t = \frac{1}{2}X_t dt + Y_t dB_t, \quad dY_t = \frac{1}{2}Y_t dt + X_t dB_t.$$

Show that the process $(X_t^2 - Y_t^2)_{t \geq 0}$ is constant.

9. Consider the diffusion process $((X_t)_{t \geq 0}, (\mathbb{P}_x)_{x \in \mathbb{R}})$ with the generator $Af(x) = x^2 f''(x) + x f'(x)$. Determine whether X is positive $\mathbb{P}_1$ -almost surely.

10. Consider the diffusion process $((X_t)_{t \geq 0}, (\mathbb{P}_x)_{x \in \mathbb{R}})$ with the generator $Af(x) = (x+1)^2 f''(x)$. Let $\tau = \inf\{t \geq 0 : X_t \in \{1, 3\}\}$. Compute $\mathbb{E}_2 \tau$.

11. Consider the process $((B_t, \int_0^t B_s ds))_{t \geq 0}$ in $\mathbb{R}^2$. Is it a diffusion? If yes, find the generator.

12. For any $t \geq 0$, let $g_t = \sup\{s \leq t : B_s = 0\}$ be the last zero of B before t . It can be shown that the process $X_t = t - g_t$, $t \geq 0$, is a time-homogeneous Markov process on $\mathbb{R}_+$, and its generator acts on $f \in C_{unif}^1$ by

$$Af(x) = \begin{cases} f'(x) + \frac{f(0) - f(x)}{2x} & \text{if } x > 0, \\ \frac{1}{2}f'(0+) & \text{if } x = 0. \end{cases}$$

Using the above information,

a) prove that the process $Y_t = t - 2g_t$, $t \geq 0$, is a martingale;

b) let $\tau = \inf\{t \geq 0 : t - g_t = 1\}$. Prove that $\mathbb{E}\tau = 2$.