

Stochastic Processes — Problems for Quiz 2

In all problems below, B stands for the standard, one-dimensional Brownian motion.

1. (M/M/ ∞ queue). We consider the queue process with an infinite number of servers, Poisson arrivals with rate a and service times which are i.i.d. exponentially distributed with parameter b . Let X_t be the number of busy serves at time t . We assume that $X_0 = 0$.

- (i) Write the q -matrix of the process X .
- (ii) Write Kolmogorov's forward equations for $p_{0n}(t)$.
- (iii) Find the differential equation for $G(t, \theta) = \sum_{n=0}^{\infty} p_{0n}(t) \theta^n$, solve it, and identify $p_{0n}(t)$.

2. Consider the continuous Markov chain on $E = \{0, 1, 2, \dots\}$, with the q -matrix

$$Q = \begin{pmatrix} -1 & 1 & 0 & 0 & 0 & \dots \\ 1 & -2 & 1 & 0 & 0 & \dots \\ 0 & 2 & -4 & 2 & 0 & \dots \\ 0 & 0 & 4 & -8 & 4 & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots \end{pmatrix}$$

- (i) For a given $i \in E$, find the time-average fraction of time spent in a state $i \in \mathbb{Z}_+$.
- (ii) For a given $i \in E$, find the expectation of the return time to i .
- (iii) For a given $i \in E$, find the expected number of transitions between each entry into state i .

3. Show that $\lim_{t \rightarrow \infty} \sqrt{t} \mathbb{P}(B_s \leq 1 \text{ for all } s \in [0, t]) = \sqrt{2/\pi}$.

4. For a given $t > 0$, find the density of the two-dimensional random variable $(B_t, \sup_{s \in [0, t]} B_s)$.

5. Let $\tau = \inf\{t \geq 0 : B_t = 3\}$ and $\sigma = \inf\{t > \tau : B_t = 0\}$. Compute $\mathbb{P}(\sigma < \tau + 1)$.

6. Consider the homogeneous continuous Markov family X on $E = \{1, 2, 3\}$, with the kernel

$$P(t) = \frac{1}{5} \begin{pmatrix} 2 + 3e^{-5t} & 1 - e^{-5t} & 2 - 2e^{-5t} \\ 2 - 2e^{-5t} & 1 + 4e^{-5t} & 2 - 2e^{-5t} \\ 2 - 2e^{-5t} & 1 - e^{-5t} & 2 + 3e^{-5t} \end{pmatrix}.$$

Let $\tau = \inf\{t : X_t = 3\}$.

- (i) Find $f_i(t) = \mathbb{P}_i(\tau \geq t)$, $i = 1, 2, 3$. *Hint: try to find Kolmogorov-type equations for f_i .*
- (ii) Compute $\mathbb{P}_1(X_4 = 2, \tau \leq 4)$ and $\mathbb{E}_2(e^{iX_{\tau+2} - iX_\tau} | \mathcal{F}_\tau^X)$.

7. A homogeneous strong Markov family $((X_t)_{t \geq 0}, (\mathbb{P}_x)_{x \in \mathbb{R}_+})$ on $\mathbb{R}_+$ has continuous trajectories and satisfies the condition $\mathbb{P}_x(\sup_{t \geq 3} X_t \geq \frac{1}{2}x) \leq \frac{1}{5}$ for all $x \in \mathbb{R}_+$. Let $\tau = \inf\{t \geq 0 : X_t \leq 1\}$. Prove that $\mathbb{P}_4(\tau \leq 6) \geq \frac{16}{25}$.

8. Let X be a homogeneous regular jump Markov process on $(E, \mathcal{E})$, started at $x_0 \in E$. Let $\sigma = \inf\{t \geq 0 : X_t \neq x_0\}$. Using the strong Markov property, show that X_σ and σ are independent.

9. Let N be a Poisson process with parameter 2 and let $\tau = \inf\{t \geq 0 : N_t > t\}$. Compute $\mathbb{P}(N_{\tau+1} < \tau + 1)$.

10. Prove that $|B|$ is a time-homogeneous Markov process and compute its transition function.

11. Let $\tau = \inf\{t \geq 0 : B_t = 0\}$ and put $B_t^\sigma := B_{\tau \wedge t}$. Find the infinitesimal generator of B^σ .

12. Let X be a homogeneous Markov process on $(E, \mathcal{E})$ with generator A .

- (i) Prove that the state-space process $Z = ((t, X_t))_{t \geq 0}$ is a Markov process on $[0, \infty) \times E$.
- (ii) Prove that the generator of Z is equal to $\frac{\partial}{\partial t} + A$ on a suitable space of functions on $[0, \infty) \times E$.

13. Let X be a homogeneous Markov process with state space E and the associated semigroup $(\mathcal{T}_t)_{t \geq 0}$. Show that for every bounded, measurable function f on E and all $t \geq 0$, the process $(\mathcal{T}_{t-s}f(X_s))_{s \in [0, t]}$ is a martingale.