Stochastic Processes — Problems for Quiz 1

1. For a fixed $p \in (0,1)$, let $\xi_1, \xi_2, \ldots$ be a sequence of independent random variables with $\mathbb{P}(\xi_j = 1) = p$, $\mathbb{P}(\xi_j = 0) = 1 - p$ for all j. Let $N = (N_t)_{t\geq 0}$ be an independent Poisson process with parameter λ. Show that the process $M_t = \sum_{n=1}^{N_t} \xi_n$, $t \geq 0$, is a Poisson process with parameter λp.

2. Let $N = (N_t)_{t\geq 0}$ be a Poisson process with parameter $\lambda > 0$. Compute $\lim_{t\to\infty} \mathbb{P}(N_t \leq \lambda t)$.

3. Let $N = (N_t)_{t\geq 0}$ be a Poisson process with intensity $\lambda > 0$ and let $\tau_6 = \inf\{t \geq 0 \colon N_t = 6\}$. Compute $\mathbb{E}(\tau_6 \mid N_1 = 3)$ and $\mathbb{P}(N_1 = 3 \mid \tau_6)$.

4. Let X be the homogeneous continuous-time Markov chain with the state space $E = \{0,1\}$ and generator Q given by

$$Q = \begin{pmatrix} -2 & 2 \\ 1 & -1 \end{pmatrix}.$$

Compute the transition function $P = (P_t)_{t\geq 0}$. Compute the distribution of X_t for a given $t > 0$, if $\mathbb{P}(X_0 = 0) = 1/3$ and $\mathbb{P}(X_0 = 1) = 2/3$.

5. Consider a machine that works for an exponential amount of time with mean $1/\lambda$ before breaking down, and suppose that it takes an exponential amount of time with mean $1/\mu$ to repair the machine. If the machine is in working condition at time 0, then what is the probability that it will be working at time $t = 10$?

6. Consider a chemical reaction during which the molecules of the compound A change irreversibly to the molecules of the compound B. Suppose the initial concentration of A is N molecules. If there are j molecules of A at time t, then each of these molecules changes to a B molecule within the time interval $(t, t+h]$ with probability $qh + o(h)$, $h \to 0$, where $q > 0$ is a given parameter. Model the number of molecules A by a homogeneous continuous-time Markov chain, find its generator Q and the transition matrix of its embedded chain.

7. The queueing system consists of a main device and a standby device. If the main device is free, then the request is processed by it. Otherwise, if the standby device is free, then the request is processed to the end by the standby device. The intensities of processing by the main and standby devices are equal to μ_1 and μ_2, respectively. The intensity of arriving requests equals λ, if both devices are busy then a new request is discarded. Characterize the Markov process describing this queueing system and compute the transition kernel.

8. Let E be a finite space and let $X = (X_t)_{t\geq 0}$, $X^{(n)} = (X_t^{(n)})_{t\geq 0}$, $n = 1, 2, \ldots$, be homogeneous Markov chains on E, with q-matrices Q and $Q^{(n)}$, respectively. Prove that if $Q^{(n)} \to Q$ entrywise, then the finite-dimensional distributions of $X^{(n)}$ converge weakly to the finite distributions of X.

9. Let α be a fixed positive number. Consider a discrete-time Markov chain on $E = \{1, 2, \ldots\}$, starting from the state 1, with the following transition probabilities: a state $k \in E$ moves in one step to state 1 with probability $(k+1)^{-\alpha}$, and to state $k+1$ with probability $1 - (k+1)^{-\alpha}$. Is the chain irreducible? For which values of α is the chain recurrent? For which values of α does a stationary distribution exist?

10. Show that for a birth–death process with rates λ_n, μ_n, the stationary probabilities (if exist) satisfy

$$\pi_n = \pi_0 \prod_{k=0}^{n-1} \frac{\lambda_k}{\mu_{k+1}}.$$

Use this to compute stationary distribution for the M/M/1 queue.

11. Consider a Markov process for which the embedded Markov chain is a birth-death chain with transition probabilities $\gamma_{i,i+1} = 2/5$, $\gamma_{i,i-1} = 3/5$ for all $i \geq 1$, $\gamma_{01} = 1$, and $\gamma_{ij} = 0$ otherwise.

(i) Find the stationary distribution $\pi = (\pi_i)_{i\geq 0}$ for the embedded chain.

(ii) Assume that the transition rate out of a state i, for $i \geq 0$, is given by $\lambda_i = 2^{-i}$. Find the transition rates $(q_{ij})_{i,j\in\mathbb{Z}_+}$ between states and show that there is no stationary distribution for the continuous chain.

(iii) Prove that the expected time between visits to any given state i is infinite.

(iv) Find the expected number of transitions between visits to any given state i.

(v) Argue that, starting from any state i, an eventual return to state i occurs with probability 1.