

Stochastic Processes – Additional Problems, list V

In all problems below, B is a standard Brownian motion in $\mathbb{R}$.

1. Fix a bounded continuous function $f : \mathbb{R} \rightarrow \mathbb{R}$ and a bounded measurable potential $V : \mathbb{R} \rightarrow [0, \infty)$. For $T > 0$ define

$$X_s = \exp \left(- \int_0^s V(B_r) dr \right) f(B_T), \quad 0 \leq s \leq T.$$

Show that there exists a deterministic function $u : [0, T] \times \mathbb{R} \rightarrow \mathbb{R}$ such that

$$\mathbb{E}_x(X_T | \mathcal{F}_s) = \exp \left(- \int_0^s V(B_r) dr \right) u(T-s, B_s) \quad \mathbb{P}_x\text{-almost surely.}$$

2. Let $\tau = \inf\{t \geq 0 : |B_t| = 1\}$ and define

$$u(x) = \mathbb{E}_x \left[\exp \left(-\lambda\tau - \mu \int_0^\tau B_s^2 ds \right) \right], \quad |x| < 1,$$

where $\lambda, \mu > 0$, and $\mathbb{E}_x$ denotes expectation for Brownian motion with $B_0 = x$. Show that u is the unique $C^2((-1, 1)) \cap C([-1, 1])$ solution of the boundary value problem

$$\frac{1}{2}u''(x) - \mu x^2 u(x) - \lambda u(x) = 0, \quad |x| < 1$$

with boundary conditions $u(\pm 1) = 1$. Moreover, prove that u is even, strictly increasing on $[0, 1]$ and satisfies $0 < u(x) \leq 2e^{-\sqrt{2\lambda}(1-|x|)}$.

3. Consider the homogeneous Itô diffusion

$$dX_t = X_t(1 - X_t) dt + \sqrt{X_t(1 - X_t)} dB_t, \quad X_0 = 1/2.$$

Can the diffusion exit the interval $(0, 1]$? Can it exit the interval $[0, 1)$? Describe the behavior of X as $t \rightarrow \infty$.

4. Find the smallest constant C such that for any finite stopping time τ we have

$$\mathbb{E}B_\tau^4 \leq C\mathbb{E}\tau^2.$$

Hint: You can assume that $\mathbb{E}\tau^2 < \infty$. Then the inequality reads $\mathbb{E}V(\tau, B_\tau) \leq 0$, where $V(t, x) = x^4 - Ct^2$. Try to find a function $U : [0, \infty) \times \mathbb{R} \rightarrow \mathbb{R}$ such that

$$U(0, 0) = 0, \quad U \geq V, \quad \frac{\partial U}{\partial t} + \frac{1}{2} \frac{\partial^2 U}{\partial x^2} = 0.$$

Such U exists only for sufficiently large C ; the least C is the desired best constant.