

Stochastic Processes — Additional Problems, list IV

1. Let n be a fixed dimension and let B be the Brownian motion in $\mathbb{R}^n$, started at the origin. Compute the expected amount of time spent by B in the unit ball of $\mathbb{R}^n$.

2. Let u be a nonnegative function of class C^2 defined on the strip $(0, T) \times \mathbb{R}$, where $0 < T \leq \infty$. Suppose that u satisfies the requirements

$$\begin{cases} \frac{\partial u}{\partial t} + \frac{\partial u}{\partial x} + \frac{1}{2} \frac{\partial^2 u}{\partial x^2} = 0 & \text{on } (0, T) \times \mathbb{R}, \\ \lim_{(t,y) \rightarrow (0,x)} u(t, y) = 0 & \text{for } x \in \mathbb{R}. \end{cases}$$

Show that $u = 0$ on $(0, T) \times \mathbb{R}$.

3. Consider the one-dimensional Itô diffusion

$$dX_t = b(X_t) dt + \sigma(X_t) dB_t, \quad X_0 = x,$$

where $b, \sigma : \mathbb{R} \rightarrow \mathbb{R}$ are bounded continuous functions satisfying $b(x) \rightarrow b_0 > 0$ as $x \rightarrow \infty$ and $\inf \sigma > 0$. Suppose that $\lim_{t \rightarrow \infty} X_t = \infty$ almost surely for any starting point. Show that

$$\lim_{t \rightarrow \infty} \frac{X_t}{t} = b_0 \quad \mathbb{P}_x\text{-almost surely.}$$

4. Consider the stochastic differential equation

$$dX_t = |X_t|^\alpha dB_t, \quad X_0 = 0.$$

(a) Prove that if $\alpha \geq 1/2$, then this equation has a unique strong solution.

(b) Prove that for $\alpha \in (0, 1/2)$ the equation has a non-zero weak solution, and hence there is no uniqueness in law or in path.

Hint for (b): Let V be a Brownian motion and let $M_t = \int_0^t |V_s|^{-\alpha} dV_s$. If $\tau_t = \inf\{s \geq 0 : \langle M \rangle_s > t\}$, then $W_t = M_{\tau_t}$ is a Brownian motion, use it to construct the weak solution.