

Stochastic Processes — Additional Problems, list III

1. Let κ, θ, σ and x be positive constants. Consider the stochastic differential equation

$$dX_t = \kappa(\theta - \ln X_t)X_t dt + \sigma X_t dB_t,$$

with the initial condition $X_0 = x$. Find an explicit formula for the strong solution.

2. Let $c : \mathbb{R} \rightarrow \mathbb{R}$ be strictly positive and continuous function on $\mathbb{R}$. Consider a Feller-Dynkin process, whose generator, when restricted to $f \in C_c^2(\mathbb{R})$, is given by

$$Af(x) = \frac{1}{2}c(x)f''(x).$$

(a) Show that if $a < x < b$ and $\tau = \inf\{t : X_t \in \{a, b\}\}$, then

$$\mathbb{E}_x \tau = \int_a^b \frac{2}{c(z)} \frac{(x \wedge z - a)(b - x \vee z)}{b - a} dz.$$

(b) Show that if $\tau = \inf\{t : X_t = a\}$, then for $x > a$ we have

$$\mathbb{E}_x \tau < \infty \quad \text{if and only if} \quad \int_0^\infty \frac{1}{c(x)} dx < \infty.$$

3. Let $X = (X_t)_{t \geq 0}$ be the Cauchy process, i.e.,

1° we have $X_0 = 0$ almost surely;

2° X has independent increments;

3° for any $0 \leq s < t$, the variable $X_t - X_s$ has the Cauchy distribution with parameter $t - s$, i.e., with density

$$g(x) = \frac{1}{\pi} \frac{t - s}{(t - s)^2 + x^2}, \quad x \in \mathbb{R}.$$

4° The trajectories of X are càdlàg almost surely.

Prove that X is a homogeneous Markov process and compute its generator.

4. For $\delta > 0$, define the square of δ -dimensional Bessel process X as the unique strong solution of the equation

$$X_t = 2 \int_0^t \sqrt{X_s} dB_s + \delta t, \quad t \geq 0.$$

Prove that $\tau = \inf\{t : X_t = 1\}$ and $(\sup_{t \leq 1} X_t)^{-1}$ have the same distribution.

Hint: search for scale-invariance properties of X .

Remark. Note that if δ is an integer, then X has the same law as $(|B_t|^2)_{t \geq 0}$, where B is the δ -dimensional Brownian motion.