

Stochastic Processes — Additional Problems, list II

1. Let $Z_0, Z_1, Z_2, \dots$ be a sequence of i.i.d. $N(0, 1)$ random variables. Show that

$$B_t = \frac{t}{\sqrt{\pi}} Z_0 + \sqrt{\frac{2}{\pi}} \sum_{m=1}^{\infty} \frac{\sin(mt)}{m} Z_m, \quad t \geq 0,$$

is a standard Brownian motion on $[0, \pi]$.

2. Let (X, Y) be a Brownian motion in $\mathbb{R}^d \times \mathbb{R}$ started at $(0, 0) \in \mathbb{R}^d \times \mathbb{R}$. For a fixed $a > 0$, let $\tau = \inf\{t > 0 : Y_t = a\}$. Find the density of the random variable X_τ .

3. Show that the probability that the standard Brownian motion has no zero in $(t, 1)$ is equal to $\frac{2}{\pi} \arcsin(\sqrt{t})$, and hence the position of the last zero preceding 1 is distributed over $(0, 1)$ with density $g(t) = \frac{1}{\pi\sqrt{t(1-t)}} \chi_{(0,1)}(t)$.

4. Let $X = (X_t)_{t \in [0,1]}$ be a Brownian bridge, i.e., a centered Gaussian process with the covariance function $\text{Cov}(X_t, X_s) = s(1-t)$ for $0 \leq s \leq t \leq 1$. Prove that for any $\alpha, \beta > 0$,

$$\mathbb{P}\left(X_t \leq \alpha(1-t) + \beta t \text{ for all } 0 \leq t \leq 1\right) = 1 - e^{-2\alpha\beta}.$$