Stochastic Processes — Additional Problems, list I

1. Let N be a Poisson process with intensity 1. Prove that with probability 1, there are infinitely many positive integers n such that $N_n = n$.

Hint: use the theory of discrete-time martingales.

2. Let $X = (X_t)_{t\geq 0}$ be a centered Gaussian process. Prove that X is a Markov process if and only if its covariance function K satisfies

$$K(s,u)K(t,t) = K(s,t)K(t,u)$$

for all $s < t < u$.

3. Let $P = (P_t)_{t\geq 0}$ be the transition kernel of a time-homogeneous continuous Markov chain, $P_t = (p_{ij}(t))_{i,j\in E}$. Prove that for each $i \neq j$, the function $t \mapsto p_{ij}(t)$ is either identically zero on $(0,\infty)$, or never vanishes on $(0,\infty)$.

4. Let $(x_t^{(n)})_{t\geq 0}$, $n = 1, 2, \ldots$, be a family of independent continuous Markov chains on the state space $\{0,1\}$, with the intensity matrix equal to

$$\begin{pmatrix} -1 & 1 \\ n^2 & -n^2 \end{pmatrix}$$

and satisfying $x_0^{(n)} = 0$. Prove that $X_t = \sum_{n=1}^{\infty} 2^{-n} x_t^{(n)}$, $t \geq 0$, defines a continuous Markov chain on the set

$$E = \Big\{ u \in [0,1] \,:\, \text{in a binary expansion, } u \text{ has a finite number of 1's} \Big\}.$$

Show that X has a standard transition function and all states of E are instantaneous.