

Stochastic Processes, Exercises - 9

1. Consider the one-dimensional Itô diffusion

$$dX_t = X_t dt + \sqrt{X_t} dB_t, \quad t \geq 0,$$

with the initial condition $X_0 = 1$. Describe the behavior of X as $t \rightarrow \infty$.

2. For fixed $\theta > 0$ and $\sigma > 0$, consider the Ornstein-Uhlenbeck process

$$dX_t = -\theta X_t dt + \sigma dB_t, \quad t \geq 0.$$

- (a) Write down the backward and forward equations for the transition density.
(b) Solve these equations.

3. Let D be an open subset of $\mathbb{R}^n$ and let $u : D \rightarrow \mathbb{R}$ be a harmonic function. Using probabilistic methods, prove that u has the mean-value property: for any $\overline{B(x, r)} \subset D$ we have

$$\frac{1}{\mu_n(\partial B(x, r))} \int_{\partial B(x, r)} u d\mu_n = u(x),$$

where μ_n is the surface measure on $\partial B(x, r)$.

4. Let $D \subset \mathbb{R}^n$ be a bounded connected open set and let $f : \partial D \rightarrow \mathbb{R}$ be a bounded continuous function. Assume further that B is a Brownian motion in $\mathbb{R}^n$ and let $\tau_D = \inf\{t \geq 0 : B_t \in \partial D\}$. Prove that τ_D is finite and the function $u(x) = \mathbb{E}_x f(B_{\tau_D})$ has the mean value property.

5. Let $u : \mathbb{R} \times [-1, 1] \rightarrow \mathbb{R}$ be a continuous bounded harmonic function. Prove that if $u(\cdot, 1)$ and $u(\cdot, -1)$ are increasing, then for any $y \in (-1, 1)$ the function $u(\cdot, y)$ is increasing.

6. Let $D = \mathbb{R} \times (0, \infty)$ be the open upper halfplane.

- a) Prove that the bounded solution to the Dirichlet problem

$$\begin{cases} \Delta u = 0 & \text{on } D, \\ u(x, 0) = f(x) & \text{if } f \in C_b(\mathbb{R}), \end{cases}$$

is given by

$$u(x, y) = \frac{1}{\pi} \int_{\mathbb{R}} \frac{yf(s)ds}{(x-s)^2 + y^2}.$$

- b) Prove that the bounded solution to the Dirichlet problem

$$\begin{cases} \Delta u + \frac{\partial u}{\partial x} = 0 & \text{on } D, \\ u(x, 0) = f(x) & \text{if } f \in C_b(\mathbb{R}^n), \end{cases}$$

is given by

$$u(x, y) = \int_0^\infty \int_{\mathbb{R}} f(s - t/2) \cdot \frac{1}{2\pi t^2} \exp\left(-\frac{(s-x)^2 + y^2}{2t}\right) ds dt.$$

- c) Prove that the Dirichlet problem

$$\begin{cases} \Delta u + \frac{\partial u}{\partial x} = 1 & \text{on } D, \\ u(x, 0) = f(x) & \text{if } f \in C_b(\mathbb{R}^n), \end{cases}$$

does not have bounded solutions.

7. Let $B = (B^{(1)}, B^{(2)})$ be a Brownian motion in $\mathbb{R}^2$ and let $\tau = \inf\{t : |B_t^{(1)}| + |B_t^{(2)}| = 1\}$. Compute $\mathbb{E}_{(0,0)} |B_\tau^{(1)}|$.