

Stochastic Processes, Exercises - 8

1. Let X be a one-dimensional diffusion started at 1, with the generator $\mathcal{A}$ satisfying

$$\mathcal{A}f(x) = xf'(x) + \frac{1}{2}x^2f''(x), \quad x \in \mathbb{R},$$

for all $f \in C_c^2(\mathbb{R})$. Let $\tau = \inf\{t \geq 0 : X_t = 5\}$. Is it true that $\tau < \infty$ with probability 1?

2. Show that the solution X of

$$dX_t = X_t^2 dt + X_t dB_t,$$

where $X_0 = 1$ and B is a standard Brownian motion, has the representation

$$X_t = \exp\left(\int_0^t X_s ds - \frac{1}{2}t + B_t\right).$$

Deduce that X does not reach zero in finite time. Show that with positive probability, X explodes in finite time.

3. Let g and h be deterministic functions of time and let X be the Itô diffusion

$$dX_t = -X_t dt + h(t) dt + g(t) dB_t.$$

Prove that X is a Gaussian process, find its mean and variance.

4. Let M be a continuous real-valued local martingale with respect to a right-continuous filtration $(\mathcal{F}_t)_{t \geq 0}$. Assume that $M_0 = 0$ and $\lim_{t \rightarrow \infty} M_t = \infty$ almost surely. For $t \geq 0$, define

$$\tau_t = \inf\{s \geq 0 : \langle M \rangle_s > t\}.$$

Prove that the process $B_t := M_{\tau_t}$ is a standard Brownian motion with respect to the filtration $(\mathcal{G}_t)_t = (\mathcal{F}_{\tau_t})_t$.