

Stochastic Processes, Exercises - 7

1. Find the strong solution to the stochastic differential equation

$$dX_t = (\mu - \alpha X_t)dt + \sigma dB_t, \quad X_0 = x.$$

2. For a fixed $a, b \in \mathbb{R}$ consider the stochastic differential equation

$$dY_t = -\frac{Y_t}{1-t}dt + dB_t, \quad 0 \leq t < 1, \quad Y_0 = 0.$$

Solve this equation and prove that Y is the Brownian bridge.

3. Consider the differential equation

$$dX_t = 3X_t^{1/3}dt + 3X_t^{2/3}dB_t, \quad X_0 = 0,$$

- a) Show that it has infinitely many strong solutions of the form

$$X_t^\theta = \begin{cases} 0 & 0 \leq t < \beta_\theta \\ B_t^3 & \beta_\theta \leq t < \infty \end{cases}$$

where $0 \leq \theta < \infty$ and $\beta_\theta = \inf\{s \geq \theta : B_s = 0\}$.

- b) Suppose that ξ is a Bernoulli random variable independent of B , assume that $\sigma(\xi) \subset \mathcal{F}_0$ and let, for $t \geq 0$,

$$X_t = \begin{cases} 0 & \text{if } \xi = 0, \\ B_t^3 & \text{if } \xi = 1. \end{cases}$$

Prove that this is a weak solution, and it is not a strong solution.

4. Let X be an Itô diffusion $dX_t = b(X_t)dt + \sigma(X_t)dB_t$, $t \geq 0$, such that b and σ are bounded on bounded sets. Prove that there exists a non-decreasing process $(\mathcal{L}_t)_{t \geq 0}$, measurable with respect to $\mathcal{F}^X$, such that

$$|X_t| = |X_0| + \int_0^t \text{sgn}(X_s)dX_s + \mathcal{L}_t, \quad t \geq 0.$$

Prove that if σ is even, then $\mathcal{L}$ is measurable with respect to $\mathcal{F}^{|X|}$. Show that if σ is bounded away from zero, then $\int_0^t \mathbf{1}_{\{X_s=0\}}ds = 0$.

5. Prove that the equation

$$dX_t = -\frac{\mathbf{1}_{\{X_t \neq 0\}}}{2X_t}dt + dB_t, \quad X_0 = 0,$$

does not have weak solutions.

6. (i) Show that the differential equation

$$dX_t = X_t^2 dt, \quad X_0 = 1,$$

has a unique strong solution that explodes almost surely in finite time.

- (ii) Show that the differential equation

$$dX_t = X_t^2 dt + X_t^{3/2} dB_t, \quad X_0 = 1,$$

has a unique strong solution that does not explode.

7. Assume that $b, \sigma : \mathbb{R}_+ \times \mathbb{R} \rightarrow \mathbb{R}$ are measurable and bounded functions. Moreover, suppose there exist a constant $L > 0$ and a strictly increasing function $\rho : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ with $\int_0^\varepsilon \rho^{-2}(x)dx = +\infty$ for any $\varepsilon > 0$ such that

$$|b(t, x) - b(t, y)| \leq L|x - y|, \quad |\sigma(t, x) - \sigma(t, y)| \leq \rho(|x - y|)$$

for all $x, y \in \mathbb{R}$, $t \in [0, T]$. Prove that the solution of the equation

$$dX_t = b(t, X_t)dt + \sigma(t, X_t)dB_t, \quad X_0 = x, \quad t \in [0, T],$$

is unique in the pathwise sense.