

Stochastic Processes, Exercises - 6

1. Let $B = (B^{(1)}, B^{(2)}, \dots, B^{(n)})$ be an n -dimensional Brownian motion started at zero and $\tau = \inf \left\{ t \geq 0 : (B_t^{(1)})^4 + (B_t^{(2)})^4 + \dots + (B_t^{(n)})^4 = 1 \right\}$. Compute

$$\mathbb{E} \int_0^\tau \left((B_t^{(1)})^2 + (B_t^{(2)})^2 + \dots + (B_t^{(n)})^2 \right) dt.$$

2. Let B be a one-dimensional Brownian motion. For any $t \geq 0$, let

$$g_t = \sup \{ s \leq t : B_s = 0 \}$$

be the last zero of B before t .

(i) Show that the process $X_t = t - g_t$, $t \geq 0$, is a time-homogeneous Markov process on $\mathbb{R}_+$. Prove that if $f : \mathbb{R}_+ \rightarrow \mathbb{R}$ is a continuous function vanishing at infinity, whose first derivative is uniformly continuous, then the generator of X acts on f by

$$Af(x) = \begin{cases} f'(x) + \frac{f(0) - f(x)}{2x} & \text{if } x > 0, \\ \frac{1}{2}f'(0+) & \text{if } x = 0. \end{cases}$$

(ii) Prove that the process $Y_t = t - 2g_t$, $t \geq 0$, is a martingale.

(iii) Let $\tau = \inf \{ t \geq 0 : t - g_t = 1 \}$. Prove that $\mathbb{E}\tau = 2$.

3. Verify that the given processes solve the corresponding stochastic differential equations.

(i) The process $X_t = e^{B_t}$ solves

$$dX_t = \frac{1}{2}X_t dt + X_t dB_t, \quad X_0 = 1.$$

(ii) The process $X_t = \frac{B_t}{1+t}$ solves

$$dX_t = -\frac{1}{1+t}X_t dt + \frac{1}{1+t}dB_t, \quad X_0 = 0.$$

(iii) The process $X = (X_t^{(1)}, X_t^{(2)}) = (t, e^t B_t)$ solves

$$dX_t = \begin{pmatrix} 1 \\ X_t^{(2)} \end{pmatrix} dt + \begin{pmatrix} 0 \\ e^{X_t^{(1)}} \end{pmatrix} dB_t, \quad X_0 = (0, 0).$$

(iv) Let $B = (B^{(1)}, B^{(2)})$ be a two-dimensional Brownian motion. Then the process $X = (X^{(1)}, X^{(2)}) = (tB_t^{(1)}, B_t^{(1)}B_t^{(2)})$ solves the equation

$$dX_t = \begin{pmatrix} \frac{X_t^{(1)}}{t} \\ 0 \end{pmatrix} dt + \begin{pmatrix} t & 0 \\ tX_t^{(2)} & X_t^{(1)} \end{pmatrix} dB_t, \quad X_0 = (0, 0).$$

(v) The process $X_t = B_t^2$ solves

$$dX_t = dt + 2\sqrt{X_t}d\tilde{B}_t,$$

where $\tilde{B}_t = \int_0^t \text{sgn}(B_s)dB_s$.