

Stochastic Processes, Exercises - 5

1. Let $\tau = \inf\{t > 0 : B_t = 0\}$. For any $x \geq 0$, compute $\mathbb{E}_x \int_0^\tau f(B_t) dt$.
2. Compute the generator of the Poisson process.
3. Let $\mu \in \mathbb{R}$ be a fixed parameter. Prove that the drifted Brownian motion $B_t^\mu = B_t + \mu t$, $t \geq 0$, is a time homogeneous Markov process and compute its generator. Is it a Feller process?

4. Compute the generator of the Ornstein-Uhlenbeck semigroup

$$\begin{aligned} \mathcal{T}_t f(x) &= \frac{1}{\sqrt{2\pi(1-e^{-2t})}} \int_{\mathbb{R}} f(y) \exp\left(-\frac{|x - e^{-t}y|^2}{2(1-e^{-2t})}\right) dy \\ &= \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} f(e^{-t}x + \sqrt{1-e^{-2t}}y) e^{-y^2/2} dy \\ &= \mathbb{E}f(e^{-t}x + \sqrt{1-e^{-2t}}B_1) \end{aligned}$$

on an appropriately regular functions.

5. Consider the Ornstein-Uhlenbeck family $((X_t)_{t \geq 0}, (\mathbb{P}_x)_{x \in \mathbb{R}})$ and let $\tau = \inf\{t : X_t \in \{0, 2\}\}$. Compute $\mathbb{P}_1(X_\tau = 2)$ and $\mathbb{E}_1 \tau$.

6. Let Z be a strictly positive random variable and define

$$\mathcal{T}_t f(x) = \mathbb{E}f(\max\{tZ, x\}), \quad f \in B(\mathbb{R}_+).$$

- (i) Prove that $(\mathcal{T}_t)_{t \geq 0}$ is a semigroup if and only if Z^{-1} is exponentially distributed: $Z^{-1} \sim \text{Exp}(\lambda)$ for some $\lambda \in (0, \infty)$.
- (ii) Suppose that $(\mathcal{T}_t)_{t \geq 0}$ is a semigroup. Write down the analytic form of $\mathcal{T}_t$ and prove that

$$D(A) \supset \{f \in C^2(\mathbb{R}_+) : f \text{ is bounded and } f'(0+) = 0\}$$

and

$$Af(x) = \lambda \int_x^\infty ((f(y) - f(x))y^{-2} dy.$$

7. a) Is there a Markov family such that the associated generator A acts on $f \in C_c^3(\mathbb{R})$ via $Af = f'''$?
- b) Suppose that a generator A of some Markov family on $\mathbb{R}^d$ acts on $C_c^2(\mathbb{R}^d)$ by

$$Af(x) = \frac{1}{2} \sum_{i,j=1}^d a_{ij}(x) \frac{\partial^2 f(x)}{\partial x_i \partial x_j},$$

where a_{ij} are continuous functions on $\mathbb{R}^d$. Prove that for each x , the matrix $(a_{ij}(x))_{i,j=1}^d$ is semipositive-definite.