

Stochastic Processes, Exercises - 4

1. Let $(X_n)_{n \geq 0}$ be a homogeneous discrete Markov chain on E with a transition matrix P . For a given state $0 \in E$, let $T_0 = \inf\{n \geq 1 : X_n = 0\}$ be the return time to 0. Prove that

$$\pi_j = \mathbb{E}_0 \left(\sum_{n=1}^{T_0} \mathbb{1}_{\{X_n=j\}} \right), \quad j \in E,$$

defines an invariant measure on E .

2. Let $(X_t)_{t \geq 0}$ be a homogeneous continuous Markov chain on E with a transition kernel P . For a given state $0 \in E$, let $R_0 = \inf\{t > 0 : X_t = 0, X_s \neq 0 \text{ for some } s \in (0, t)\}$ be the return time to 0. Prove that

$$\nu_j = \mathbb{E}_0 \left(\int_0^{R_0} \mathbb{1}_{\{X_t=j\}} dt \right), \quad j \in E,$$

defines an invariant measure on E .

3. For a standard Brownian motion B , let $\tau = \inf\{t \geq 0 : B_t = 5\}$. Compute $\mathbb{P}(B_{\tau+2} \geq 5)$ and $\mathbb{E}B_{\tau+2}^2$.

4. Let N be a Poisson process with intensity λ and let $X_t = (-1)^{N_t}$. Compute

$$\mathbb{P}(X_1 = 1, \text{ and } X_t = -1 \text{ for some } t \in (0, 1)).$$

5. For the Wiener family $((B_t)_{t \geq 0}, (\mathbb{P}_x)_{x \in \mathbb{R}})$, derive

$$\mathbb{P}_x \left(\inf_{u \in [0, t]} B(u) \leq 0 \right) \quad \text{and} \quad \mathbb{P}_x \left(B(t) > y, \inf_{u \in [0, t]} B(u) > 0 \right).$$

6. Let $f, g : \mathbb{R} \rightarrow \mathbb{R}$ be bounded integrable Borel functions with $\int_{\mathbb{R}} g(x) dx \neq 0$. Show that

$$\lim_{T \rightarrow \infty} \frac{\int_0^T f(B_t) dt}{\int_0^T g(B_t) dt} = \frac{\int_{\mathbb{R}} f(x) dx}{\int_{\mathbb{R}} g(x) dx}.$$

7. Let $((B_t)_{t \geq 0}, (\mathbb{P}_x)_{x \in \mathbb{R}})$ be a homogeneous Wiener family. Moreover, let $\tau := \inf\{t \geq 0 : B_t = 0\}$, and define $B_t^o := B_{\tau \wedge t}$. Show that $((B_t^o)_{t \geq 0}, (\mathbb{P}_x)_{x \in \mathbb{R}_+})$ is a homogeneous Markov family and find its transition function.