Stochastic Processes, Exercises - 3

1. Let $0 < p = 1 - q < 1$. Let X be the homogeneous continuous-time Markov chain with the state space $E = \mathbb{Z}_+$ and the generator Q given by

$$Q = \begin{pmatrix} -1 & 1 & 0 & 0 & \cdots \\ pq & -p & p^2 & 0 & \cdots \\ p^2q & 0 & -p^2 & p^3 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

Decide whether the stationary distribution of X exists or not; and if it does exist, find it. Solve the same problem for the matrix

$$Q = \begin{pmatrix} -1 & 1 & 0 & 0 & 0 & \dots \\ 1 & -3 & 2 & 0 & 0 & \dots \\ 1 & 0 & -4 & 3 & 0 & \dots \\ 1 & 0 & 0 & -5 & 4 & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots \end{pmatrix}$$

2. Requests of two types A and B arrive at the device at random with intensities λ_1, λ_2 and are processed with intensities μ_1, μ_2, respectively. If the device is busy, then any other pending request at the moment is discarded.

(i) Define the Markov process describing such a queueing system. Write the Kolmogorov equations for the transition probabilities.

(ii) Find the probability that at some distant time moment, an arriving request will be discarded.

3. Let $p \in (0, 1)$ and set $q = 1 - p$. Consider the discrete-time Markov chain Y with state space $\{0, 1, 2, \dots\}$ and transition matrix

$$\Gamma = \begin{pmatrix} 0 & 1 & 0 & 0 & \dots \\ q & 0 & p & 0 & \dots \\ q & 0 & 0 & p & \dots \\ q & 0 & 0 & 0 & \dots \\ \dots & \dots & \dots & \dots & \dots \end{pmatrix}.$$

(i) Show that each state is positive recurrent: we have $\mathbb{E}(T_1^{(i)} \mid Y_0 = i) < \infty$ for each i, where $T_1^{(i)} = \inf\{n \geq 1 : Y_n = i\}$.

(ii) Suppose that holding parameters are given by $\lambda_n = p^n$. Show that the corresponding continuous Markov chain is not positive recurrent.

4. For given λ, $\mu > 0$, consider the homogeneous continuous Markov chain on $E = \mathbb{Z}_+$ with the q-matrix

$$Q = \begin{pmatrix} -\lambda & \lambda & 0 & 0 & \dots & \\ \mu & -(\mu + \lambda/2) & \lambda/2 & 0 & 0 & \dots \\ 0 & \mu & -(\mu + \lambda/3) & \lambda/3 & 0 & \dots \\ 0 & 0 & \mu & -(\mu + \lambda/4) & \lambda/4 & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots \end{pmatrix}.$$

(i) For any $i \in \mathbb{Z}_+$, find the time-average fraction of time spent in a state $i \in \mathbb{Z}_+$.

(ii) For each i, find both the time-average interval and the time-average number of overall state transitions between successive visits to i.