

## Stochastic Processes, Exercises - 2

1. Let  $X = (X_t)_{t \geq 0}$  be a continuous Markov chain on  $E = \{1, 2, 3\}$  with

$$Q = \begin{pmatrix} -2 & 2 & 0 \\ 1 & -3 & 2 \\ 0 & 2 & -2 \end{pmatrix} ..$$

Let  $\tau_n$  be the moment of  $n$ -th jump of the process  $X$ .

- (a) Compute  $\mathbb{P}(X_{\tau_1} = 1, X_{\tau_2} = 2, X_{\tau_3} = 3 \mid X_0 = 2)$ .
- (b) Compute the transition function  $P = (P_t)_{t \geq 0}$ .
- (c) Solve (b) for the matrix

$$Q = \begin{pmatrix} -3 & 1 & 2 \\ 2 & -3 & 1 \\ 1 & 1 & -2 \end{pmatrix}.$$

2. Consider a liquid solution of two chemical compounds  $A$  and  $B$ . The chemical reaction is described as follows:  $A + B \rightarrow C$ , i.e. one  $A$ -molecule reacts with one  $B$ -molecule and they produce one  $C$ -molecule. Let the initial concentrations of the compounds  $A$ ,  $B$ , and  $C$  be  $a$ ,  $b$ , and  $0$  molecules, respectively, and set  $N := \min\{a, b\}$ . If there are  $j$  molecules of  $C$  at time  $t$ , then the reaction will produce exactly one molecule of the compound  $C$  within the time interval  $(t, t + h]$  with probability  $q(a - j)(b - j)h + o(h)$ ,  $h \rightarrow 0+$  for  $j = 0, 1, 2, \dots, N$ . Model the concentration of  $C$  at time  $t$  by a homogeneous continuous-time Markov chain  $X = (X_t)_{t \geq 0}$ , find its generator and the transition matrix of its embedded chain.

3. For a pure birth process, write down the Kolmogorov forward and backward equations. Compute  $p_{0n}(t)$  for any  $n$  and  $t$ . Find the necessary and sufficient conditions for non-explosion.

4. Let  $X = (X_t)_{t \geq 0}$  be a birth-and-death process with parameters  $\lambda_n > 0$  and  $\mu_n > 0$ .

- (a) Find the average time required to reach state  $n$  when starting from state  $0$ .
- (b) Show that the state  $0$  is recurrent if and only if  $\sum_{n=0}^{\infty} \prod_{j=0}^n \frac{\mu_j}{\lambda_j} = +\infty$ .
- (c) Let  $\lambda, \mu > 0$  be fixed. Show that the birth-and-death process with parameters  $n\lambda$  and  $n\mu$ , respectively, is non-explosive.

5. Consider the linear birth and death process (the intensities are  $\mu_n = n\mu$  and  $\lambda_n = n\lambda$ ). Let

$$G_1(\theta, t) = \sum_{n=0}^{\infty} p_{1n}(t)\theta^n, \quad G_2(\theta, t) = \sum_{n=0}^{\infty} p_{2n}(t)\theta^n.$$

- (a) Show that  $G_2(\theta, t) = (G_1(\theta, t))^2$ .
- (b) Write the backward equations for  $p_{1n}(t)$ ,  $n \geq 1$ , and  $p_{10}(t)$ .
- (c) Show that  $\frac{\partial G_1}{\partial t} = -(\lambda + \mu)G_1 + \lambda G_1^2 + \mu$ .
- (d) Show that for  $\lambda \neq \mu$  we have  $\frac{\partial}{\partial t} \log\left(\frac{\lambda G_1 - \mu}{G_1 - 1}\right) = \mu - \lambda$ .
- (e) Show that for  $\lambda \neq \mu$  we have  $G_1(\theta, t) = \frac{\mu(1 - \theta) - (\mu - \lambda\theta)e^{(\mu - \lambda)t}}{\lambda(1 - \theta) - (\mu - \lambda\theta)e^{(\mu - \lambda)t}}$ .
- (f) Show that for  $\lambda \neq \mu$  we have  $p_{10}(t) = \frac{\mu - \mu e^{(\mu - \lambda)t}}{\lambda - \mu e^{(\mu - \lambda)t}}$ , and find  $\lim_{t \rightarrow \infty} p_{10}(t)$ .
- (g) Show that for  $\lambda = \mu$ ,  $p_{10}(t) = \frac{\lambda t}{1 + \lambda t}$ .
- (h) By (f), for  $\mu > \lambda$  the process will reach state  $0$  (the population dies out) at some time. Let  $T_0$  be the waiting time for this event (with  $X_0 = 1$ ). Find the distribution and the mean of  $T_0$ .
- (i) Show that for  $\mu = \lambda$ ,  $G_1(\theta, t) = \frac{\lambda t + \theta(1 - \lambda t)}{1 + \lambda t - \theta \lambda t}$ , and  $p_{1n}(t) = \frac{1}{n!} \frac{(\lambda t)^{n-1}}{(1 + \lambda t)^{n+1}}$ ,  $n \geq 1$ .
- (j) Calculate  $p_{n0}(t)$ ,  $n \geq 2$ .
- (k) Let  $F_m$  be the distribution function for  $T_0^{(m)}$ , where  $T_0^{(m)}$  is the waiting time for the extinction when the process starts with  $m$  individuals. Show that

$$\lim_{m \rightarrow \infty} F_m(t) = \exp(-t^{-1}).$$