

Stochastic Processes, Exercises - 10

1. Let $T > 0$, $f \in C_b(\mathbb{R}^d)$, $g \in C_b([0, T] \times \mathbb{R}^d)$ and $\rho \in \mathbb{R}$. Let $u : [0, T] \times \mathbb{R}^d \rightarrow \mathbb{R}$ be the bounded solution of

$$\frac{\partial u}{\partial t} = \rho u + \frac{1}{2} \Delta u + g, \quad t \geq 0,$$

with the initial condition $u(0, x) = f(x)$. Give the probabilistic and analytic representation and prove that the solution is unique. Show that the problem is monotone: if $f_1 \leq f_2$, then $u_1 \leq u_2$.

2. For $a > 0$ define the hitting time $\tau_a = \inf\{t \geq 0 : B_t = a\}$. Using the Feynman-Kac formula, compute explicitly $u(x) = \mathbb{E}_x[e^{-\lambda \tau_a}]$ for $x \leq a$ and $\lambda > 0$.

3. Let $(B_t)_{t \geq 0}$ be a standard Brownian motion and consider

$$dX_t = -X_t dt + \sqrt{1 + X_t^2} dB_t, \quad X_0 = x.$$

a) Find the infinitesimal generator L .

b) Show that $M_t = X_t^2 - \int_0^t (1 + X_s^2) ds$, $t \geq 0$, is a martingale.

c) Compute $\mathbb{E}_x X_t^2$, by solving an appropriate differential equation.

d) For $a > 0$, let $\tau_a = \inf\{t \geq 0 : |X_t| \geq a\}$ and define $u(x) = \mathbb{E}_x \tau_a$. Show that u is the bounded solution to the differential equation

$$-xu'(x) + \frac{1+x^2}{2}u''(x) = -1, \quad |x| < a,$$

with boundary conditions $u(-a) = u(a) = 0$.

e) Solve the above boundary value problem.

4. Show that the solution $u(t, x)$ of the initial value problem

$$\begin{cases} \frac{\partial u}{\partial t} = \frac{1}{2}\beta^2 x^2 \frac{\partial^2 u}{\partial x^2} + \alpha x \frac{\partial u}{\partial x} & \text{for } t > 0, x \in \mathbb{R}, \\ u(0, x) = f(x), \end{cases}$$

where $f \in C_c^2(\mathbb{R})$ is given, can be expressed as follows:

$$\begin{aligned} u(t, x) &= \mathbb{E} \left[f \left(x \exp \left\{ \beta B_t + \left(\alpha - \frac{1}{2}\beta^2 \right) t \right\} \right) \right] \\ &= \frac{1}{\sqrt{2\pi t}} \int_{\mathbb{R}} f \left(x \exp \left\{ \beta y + \left(\alpha - \frac{1}{2}\beta^2 \right) t \right\} \right) \exp \left(-\frac{y^2}{2t} \right) dy. \end{aligned}$$

Prove that if $f_1 \leq f_2$, then the corresponding solutions u_1, u_2 satisfy $u_1 \leq u_2$.

5. Using the Feynman-Kac formula, prove that

$$\mathbb{E}_0 \left[\exp \left(-\lambda \int_0^t B_s^2 ds \right) \right] = \frac{1}{\sqrt{\cosh(\sqrt{2\lambda} t)}}.$$