**Stochastic Processes, Exercises - 1**

**1.** Let $E$ be at most countable. Prove that $(X_t)_{t\geq 0}$ is a continuous Markov chain on $E$ if and only if for any $0 \leq t_0 < t_1 < t_2 < \ldots < t_n$ and any $i_0,\, i_1,\, i_2,\, \ldots,\, i_n \in E$,

$$\mathbb{P}\Big(X_{t_n} = i_n \mid X_{t_{n-1}} = i_{n-1}, \ldots, X_{t_0} = i_0\Big) = \mathbb{P}\Big(X_{t_n} = i_n \mid X_{t_{n-1}} = i_{n-1}\Big).$$

**2.** Let $N = (N_t)_{t\geq 0}$ be a Poisson process with intensity $\lambda$. Prove that $X_t = (-1)^{N_t}$, $t \geq 0$, is a homogeneous Markov chain and compute its transition function.

**3.** Let $\xi$, $\eta$ be two independent random variables with the uniform distribution on $[0,1]$. Prove that the process $X_t = \mathbb{1}_{\{\xi\leq t\}} + \mathbb{1}_{\{\eta\leq t\}}$, $t \geq 0$, is a continuous Markov chain and compute its transition function. Is the chain homogeneous?

**4.** On the interval $[0,T]$ place $n$ points independently and uniformly. For the disjoint intervals $[a_1,b_1]$, $[a_2,b_2]$, $\ldots$, $[a_k,b_k] \subset [0,T]$, compute the distribution of $(X_1,\, X_2,\, \ldots,\, X_k)$, where $X_j$ is the number of points in $[a_j,b_j]$. Compute the limit of such distributions, when $n$ and $T$ tend simultaneously to infinity in such a way that $n/T$ remains constant ($= \lambda > 0$) ($k$, $a_i$ and $b_i$ remain fixed).

**5.** Let $X = (X_t)_{t\geq 0}$ be a continuous Markov chain with right-continuous paths, satisfying $X_0 = x_0$. Let $\tau = \inf\{t \geq 0 : X_t \neq x_0\}$. Prove that $\tau$ is exponentially distributed.

**6.** Let $N = (N_t)_{t\geq 0}$ be a Poisson process with intensity $\lambda$. Show that with probability 1, all jumps of $N$ have size 1.

**7.** Let $\theta$ have a Poisson distribution with parameter $\lambda$, and let $\xi_1, \xi_2, \ldots$ be independent random variables with the uniform distribution on $[0,1]$, independent of $\theta$. Show that the process $(X_t)_{t\in[0,1]}$ defined by the formula

$$X_t := \sum_{j=1}^{\theta} \mathbb{1}_{\{\xi_j\leq t\}}$$

is a Poisson process with parameter $\lambda$.