**Optimal Control, Homework III.**

*Deadline: 22.02.2024.*

**1.** Using the theory of optimal control, prove that for any $a_1, a_2, \ldots, a_n \in \mathbb{R}$ we have the inequality

$$\sum_{n=1}^{N} 2^n a_n \leq \frac{2^{n+1}}{\sqrt{3}} \left( \sum_{n=1}^{N} a_n^2 \right)^{1/2}.$$

**2.** Let $X_1, X_2, \ldots$ be independent variables with the law $Exp(1)$. Solve the optimal stopping problems

$$V = \inf_{\tau} \mathbb{E} X_\tau, \qquad V^N = \inf_{\tau \leq N} \mathbb{E} X_\tau.$$

**3.** Consider the problem

$$\sup \mathbb{E} \left\{ -\exp(-X_N) - \sum_{m=1}^{N} \exp(-u_m) \right\},$$

where $u_n$ are controls taking values anywhere in $\mathbb{R}$ and where

$$X_{n+1} = 2X_n - u_{n+1} + V_{n+1}, \qquad X_0 \text{ given.}$$

Here $V_{n+1}$, $n = 0, 1, 2, ..., N-1$, are identically and independently distributed, and $K := \mathbb{E} \exp(-V_{n+1}) < \infty$.

**4.** Using the theory of optimal control, find the best constant $C$ in the inequality

$$\int_0^\infty |f(t)| \mathrm{d}t \leq C \left( \int_0^\infty f^2(u) \mathrm{d}u + \int_0^\infty f^2(u) u^2 \mathrm{d}u \right)^{1/2},$$

to be valid for all measurable $f : [0, \infty) \to \mathbb{R}$.

**5.** Solve the problem

$$\sup_u \left\{ - \int_0^1 u^2 \mathrm{d}t + x(1) \right\},$$

where $\dot{x} = u > 0$ , $x(0) = x_0 > 0$.

**6.** Solve the problem

$$\sup_u \mathbb{E} \int_0^\infty (\sqrt{X_t} + \sqrt{u_t}) e^{-t} \mathrm{d}t,$$

where $X_0 = x_0 > 0$, $dX_t = (X_t - u_t)dt + X_t dW_t$ and the controls $u_t$ are assumed to be positive.