Optimal Control, Homework II.

Deadline: 25.01.2024. ***Solve four problems from the list below.***

1. Let ξ_1, ξ_2, …, ξ_N be a sequence of independent random variables with the same distribution $\mathbb{P}(\xi_j = 0) = \frac{2}{3}$, $\mathbb{P}(\xi_j = 1) = \frac{1}{3}$. Let $X_0 \in \mathbb{R}$ be an arbitrary number and let

$$X_{n+1} = \left(X_n - \frac{1}{2}\right)\xi_{n+1}, \qquad n = 0, 1, 2, \dots, N-1.$$

Solve the optimal stopping problem $\sup_{\tau \leq N} \mathbb{E}X_\tau$, where the supremum is taken over all stopping times adapted to the filtration generated by ξ_1, ξ_2, …, ξ_N.

Easier version: solve this problem for $N = 3$.

2. Let X_1, X_2, … be independent random variables with the distribution given by $\mathbb{P}(X_j = k) = 2^{-k-1}$, $k = 0, 1, 2, \dots$. Solve the optimal stopping problem

$$\sup_\tau \mathbb{E}\left(X_\tau - \frac{\tau}{2024}\right),$$

where the supremum is taken over all finite stopping times adapted to the natural filtration of the sequence $(X_n)_{n\geq 1}$.

3. Let $(S_n)_{n\geq 0}$ be a symmetric random walk over the integers and let $\beta \in (0,1)$ be a fixed parameter. Solve the optimal stopping problem

$$V(x) = \sup_\tau \mathbb{E}_x\left[\beta^\tau (1 - \exp(S_\tau))^+\right], \qquad x \in \mathbb{Z}$$

(here, as usual, $\mathbb{P}_x$ is the probability under which S_0 starts from x, and $\mathbb{E}_x$ is the associated expectation).

4. Solve the optimal control problem

$$\sup_u \left\{\int_0^T \sqrt[3]{u(t)}\mathrm{d}t + \sqrt[3]{x(T)}\right\},$$

where $x(0) = x_0 > 0$ and $\dot{x} = -u < 0$.

5. Find the best constant C in the inequality

$$\int_0^1 \exp\left(\left|\frac{1}{t}\int_0^t f\right|\right)\mathrm{d}t \leq C\int_0^1 \exp(|f|)\mathrm{d}t,$$

to be valid for all piecewise continuous functions f on $[0,1]$.

6. Let X_1, X_2, …, X_N be independent random variables with the uniform distribution on $[0,1]$. Solve the optimal stopping problem

$$\sup \mathbb{P}\left(X_\tau = \max_{1\leq n\leq N} X_n\right),$$

where the supremum is taken over all stopping times τ adapted to the natural filtration of X_1, X_2, …, X_N.

Easier version: solve this problem for $N = 3$.