Optimal Control, Homework I.

Deadline: 30.11.2023. In the solutions, please make use of the theory of optimal control.

1. Prove that for any nonincreasing sequence $(x_n)_{n\geq 0}$ of positive numbers we have the inequality

$$\sum_{n=0}^{\infty} \frac{x_n^2}{x_{n+1}} \geq 4x_0$$

and the constant 4 is the best possible.

2. Prove that for any nonnegative numbers a_1, a_2, ..., a_N satisfying $x = a_1 + a_2 + \ldots + a_N \leq 1$ we have

$$\frac{a_1}{1+a_1^2} + \frac{a_2}{1+a_2^2} + \ldots + \frac{a_N}{1+a_N^2} \leq \frac{N^2 x}{N^2 + x^2}.$$

3. Consider the problem

$$\sup \mathbb{E} \left\{ \sum_{m=1}^{N} (1-u_m) X_{m-1} \right\},$$

where $X_0 = 1$, $u_m \in [0,1]$ and

$$X_m = X_{m-1} + u_m X_{m-1} + \xi_m, \qquad m = 1,\, 2,\, \ldots,\, N.$$

Here $(\xi_m)_{m=1}^N$ is a sequence of independent random variables such that for any m, ξ_m has exponential distribution with parameter $1/X_{m-1}$.

Compute the above supremum and identify the optimal controls.

4. Consider the optimal control problem

$$\sup_u \mathbb{E} \sum_{m=1}^{\infty} \sqrt[3]{u_m},$$

where the process $(X_n)_{n\geq 0}$ satisfies $X_0 = 1$ and

$$X_m = (X_{m-1} - u_m)\xi_m, \qquad m = 1,\, 2,\, \ldots.$$

Here $u_m \in [0, X_{m-1}]$ and $(\xi_m)_{m\geq 1}$ is a sequence of independent, identically distributed random variables, $\mathbb{P}(\xi_m = 0) = 3/4$, $\mathbb{P}(\xi_m = 1) = 1/4$.

Compute the above supremum and identify the optimal controls.