Bellman functions, Homework II. Deadline: 23.01.2026

1. Let f, g be martingales such that $\|f\|_\infty \leq 1$ and g is differentially subordinate to f. Find the best constant in the inequality

$$\mathbb{P}(g_n \geq 1) \leq C,$$

by identifying the smallest Bellman function leading to this estimate.

2. Suppose that $p > 2$. Find the best constant C_p in the inequality

$$\|Y\|_p \leq C_p \|X\|_p,$$

where X, Y are continuous-time martingales such that X is nonnegative and Y is differentially subordinate to X.

3. Suppose that f is a function on the unit circle $\mathbb{T}$, satisfying $\|f\|_{L^\infty(\mathbb{T})} \leq 1$. For any $\lambda > 0$, identify the best constant C_λ in the estimate

$$\int_{-\pi}^{\pi} \exp(\lambda \mathcal{H}^{\mathbb{T}} f(s)) \mathrm{d}s \leq C_\lambda.$$

4. For any $K > 0$, find the smallest constant $L = L(K)$ with the following property. If $(f_n)_{n \geq 0}$ is a nonnegative martingale, then

$$\mathbb{E} f_n^* \leq K \mathbb{E}(f_n + 1) \log(f_n + 1) + L(K).$$

5*. Let $\mathcal{D}$ denote the collection of all dyadic subintervals of the unit interval $[0,1]$. Find the best absolute constant C such that the following holds. If $(\alpha_Q)_{Q \in \mathcal{D}}$ is a sequence of nonnegative numbers satisfying

$$\sum_{Q \subset R,\, Q \in \mathcal{D}} \alpha_Q \leq |R|, \qquad \text{for all } R \in \mathcal{D},$$

(where $|R|$ is the Lebesgue measure of R), then for any integrable function $f : [0,1] \to \mathbb{R}$ we have

$$\sum_{Q \in \mathcal{D}} \frac{\alpha_Q}{|Q|} \int_Q f \mathrm{d}x \leq C \|f\|_{L^2(0,1)}.$$