Bellman functions, Homework I. Deadline: 05.12.2025

1. Using the dynamic programming, prove that for any positive numbers $a_1, a_2, \dots, a_n$ satisfying $a_1 a_2 \dots a_n = 1$ the following inequality holds:

$$\frac{1}{n-1+a_1} + \frac{1}{n-1+a_2} + \dots + \frac{1}{n-1+a_n} \le 1.$$

2. Find the best constant C in the inequality

$$\int_0^\infty \exp\left(\frac{1}{t}\int_0^t f\right) \le C \int_0^\infty \exp(f),$$

to be valid for all locally integrable f on $\mathbb{R}_+$.

3. From an urn containing N balls numbered from 1 to N, we draw balls (one at a time) with replacement. The cost of each draw is equal to a given parameter $c > 0$. Let X_n denote the number of distinct balls observed in the first n draws, $n = 0, 1, 2, \dots$. Solve the problem

$$V = \sup_\tau \mathbb{E}(X_\tau - c\tau),$$

where the supremum is taken over all integrable stopping times τ.

4. Let $S = (S_n)_{n \ge 0}$ be a symmetric random walk over integers and let $\beta \in (0, 1)$ be a fixed parameter. Solve the problem

$$V(x) = \sup_\tau \mathbb{E}_x\left[\beta^\tau (1 - \exp(S_\tau))^+\right], \qquad x \in \mathbb{Z},$$

where te supremum is taken over all finite stopping times τ (adapted to the natural filtration of S).

5*. Using the Bellman function method, find the explicit value of

$$\inf\left\{a_1^p + \frac{a_2^p}{a_1} + \frac{a_3^p}{a_1 a_2} + \ldots + \frac{a_n^p}{a_1 a_2 \ldots a_{n-1}}\right\},$$

where the infimum is taken over all a_1, a_2, $\dots$, $a_n > 0$ satisfying $a_1 a_2 \dots a_n = x$.