CHAPTER 1

# Optimal Control in Deterministic Case

Our starting point is the discussion on the special, deterministic case of the theory of optimal stochastic control. We start with the analysis of discrete-time systems.

## 1. Discrete-time systems, finite horizon

The main tool used in this case is the so-called dynamic programming, an algorithm which enables to solve a certain class of problems, by an induction argument which reduces them to simpler sub-problems. Or, to put it in the reverse direction, the approach allows to tackle difficult problems by solving simpler ones first and relating these solutions to the harder context by intrinsic recurrence relations. It plays an important role in computer science, as it can be used to construct effective algorithms of polynomial complexity. Furthermore, the method is utilized in optimal planning problems (e.g. in problems of the optimal distribution of resources available, the theory of inventory management, replacement of equipment, etc.).

The basic idea can be formulated as follows. Suppose that a given system $\mathcal{S}$, taking values in some set $E$, is controlled with a procedure which consists of $N$ steps, where $N$ is a fixed positive integer. At the beginning, the system is in some state $x_0 \in E$. At the $m$-th step of the procedure, there is a possibility of applying a class of controls, each of which transforms the state $x_{m-1} \in E$ obtained through the previous operations into some new state $x_m \in E$. Formally, if $u_m$ denotes the control applied at $m$-th step, then we have the identity

$$x_m = f_m(x_{m-1}, u_m) \tag{1.1}$$

for some $f_m$ (the "transition function associated with $m$-th step"). An important feature of the method is the absence of after-effects, i.e. the controls selected for a given step may only affect the state of the system at that moment. We also emphasize that the class of controls applicable at $m$-th step may depend on the value of $x_{m-1}$: we have $u_m \in U(x_{m-1})$ for some function $U$. As the result of the whole procedure $u_1$, $u_2$, $\ldots$, $u_N$, the system is converted from the state $x_0$ into the final state $x_N$. Now, given the initial state $x_0$ and a fixed objective function

$$J^N(x_0, u_1, u_2, \ldots, u_N) = \sum_{m=1}^{N} g_m(x_{m-1}, u_m) + r(x_N), \tag{1.2}$$

there are two aspects worth investigating:

• to identify the controls $u_1^*$, $u_2^*$, $\ldots$, $u_N^*$ (if exist), which yield the optimal performance, i.e., such that we have

$$J^N(x_0, u_1^*, u_2^*, \ldots, u_N^*) = \sup_{u_1, u_2, \ldots, u_N} J^N(x_0, u_1, u_2, \ldots, u_N).$$

• to compute the explicit value $\sup_{u_1,u_2,\dots,u_N} J^N(x_0,u_1,u_2,\dots,u_N)$.

Sometimes, depending on the context, we might be interesting in the analysis of only one of the above aspects: we will illustrate this on some later examples.

Conventional methods of tackling such problems are often either inapplicable, or involve lengthy and elaborate calculations. Dynamic programming allows to solve the above problem by the following recursive formula.

THEOREM 1.1. *Under the above notation, define measurable functions $B_N$, $B_{N-1}$, …, $B_0$ on $E$ by $B_N(x) = r(x)$ and the Bellman equation*

$$B_{n-1}(x) = \sup_{u\in U(x)} \big(g_n(x,u) + B_n(f_n(x,u))\big), \quad n = N,\, N-1,\, \dots,\, 1. \tag{1.3}$$

*Then for any strategy $u_1$, $u_2$, …, $u_N$ we have $J^N(x_0,u_1,u_2,\dots,u_N) \le B_0(x_0)$. Furthermore, if for any $n$ and any $x\in E$ there is a control $\hat{u} = \hat{u}_n(x)$ for which*

$$B_{n-1}(x) = g_n(x,\hat{u}) + B_n(f_n(x,\hat{u}))\big),$$

*then the optimal strategy is given by $u_n^* = \hat{u}_n(x_{n-1})$, $n = 1, 2, \dots, N$.*

PROOF. Consider the subsystem which starts at time $n$ and then evolves according to (1.1) for $m = n+1$, $n+2$, …, $N$. Introduce the truncated functional

$$J_n^N(x_n,u_{n+1},u_{n+2},\dots,u_N) = \sum_{m=n+1}^{N} g_m(x_{m-1},u_m) + r(x_N).$$

Then, as we will show inductively, we have $J_n^N(x_n,u_{n+1},u_{n+2},\dots,u_N) \le B_n(x_n)$. Indeed, we have $J_N^N(x_N) = r(x_N) = B_N(x_N)$ and the induction step follows from

$$\begin{aligned} &J_{n-1}^N(x_{n-1},u_n,u_{n+1},\dots,u_N)\\ &\qquad = g_n(x_{n-1},u_n) + J_n^N(f_n(x_{n-1},u_n),u_{n+1},\dots,u_N)\\ &\qquad \le g_n(x_{n-1},u_n) + B_n(f_n(x_{n-1},u_n)) \le B_{n-1}(x_{n-1}). \end{aligned}$$

Plugging $n = 1$, we get the first part of the assertion. To get the second part, note that for the above special controls $u_1^*$, $u_2^*$, …, $u_N^*$, all the above inequalities become equalities. □

Several helpful comments are in order.

**I.** It is worth to rewrite the proof of $J^N(x_0,u_1,u_2,\dots,u_N) \le B_0(x_0)$ in the form

$$\begin{aligned} B_0(x_0) &\ge g_1(x_0,u_1) + B_1(x_1)\\ &\ge g_1(x_0,u_1) + g_2(x_1,u_2)B_2(x_2)\\ &\ge \dots \ge \sum_{m=1}^{n} g_m(x_{m-1},u_m) + B_N(x_N) = J^N(x_0,u_1,u_2,\dots,u_N). \end{aligned}$$

**II.** It follows from the above proof that the Bellman sequence admits the alternative definition

$$B_n(x) = \sup_{u_{n+1},u_{n+2},\dots,u_N} \Big\{ J_n^N(x_n,u_{n+1},u_{n+2},\dots,u_N) \,\big|\, x_n = x \Big\}$$

for $n = 0, 1, 2, \dots, N$. Thus in particular $B_0(x_0)$ is the desired optimal value of the function $J^N$. As we will see later, writing down the above abstract formula for $B_n$ is a convenient start for the analysis.

**III.** The above argumentation leads to the following optimality principle: given $x_0$, if there is an optimal strategy $u_1^*$, $u_2^*$, …, $u_N^*$ for $J^N$, then

· for any $n$, the strategy $u_{n+1}^*$, $u_{n+2}^*$, …, $u_N^*$ must be optimal for the functional $J_n^N(x_n, u_{n+1}, u_{n+2}, \dots, u_N)$ (where $x_n$ comes from $x_0$ by applying $u_1^*$, $u_2^*$, …, $u_n^*$);

· for any $n$, the strategy $u_1^*$, $u_2^*$, …, $u_n^*$ must be optimal for the functional

$$J^n(x_0, u_1, u_2, \dots, u_n) = \sum_{m=1}^{n} g_m(x_{m-1}, u_m) + B_n(x_n).$$

**IV.** The above discussion concerns the case in which we are interested in the largest value of $J^N$. Sometimes one wants to minimize $J^N$: then all the argumentation works, we only need to replace suprema by infima in appropriate places.

**V.** The form of the Bellman equation (1.3) is strictly connected to the additive form of the functional. As we shall see later, the general approach extends to the case of other functionals, for which the associated Bellman equations look differently.

The above statements give a transparent method of handling the problem. We solve the system (1.3) of functional equations, obtaining $B_0(x_0)$, the value of the optimal performance under the assumption that the system is initially in the state $x_0$. We also identify the optimal controls $u_1^*$, $u_2^*$, …, $u_N^*$: these are the parameters for which the suprema in (1.3) are attained. Specifically, having determined $B_0$, $B_1$, …, $B_N$ (called the Bellman sequence in the sequel), we find $u_1^*$ by the requirement

$$B_0(x_0) = g_1(x_0, u_1^*) + B_1(f_1(x_0, u_1^*)).$$

Let $x_1 = f_1(x_0, u_1^*)$ be the position of the system after the optimal first move. Then we get the control $u_2^*$ from the equation

$$B_1(x_1) = g_2(x_1, u_2^*) + B_2(f_2(x_1, u_2^*)),$$

and so on. Summarizing, the approach rests on solving the initial problem by embedding it into a class of similar sub-problems, the collection of which can be treated, as a whole, by means of the recursive formulas. We should point out, however, that still, in many cases, the analysis of the obtained setting can be technically involved and require plenty of laborious computations.

In what follows, we will see the above reasoning in various disguises. We have purposefully restrained ourselves from the discussion concerning the state space or the class of feasible controls; this would probably complicate the above presentation, as these objects can be multidimensional or change from step to step. Sometimes it will be convenient to enumerate the steps by numbers 1, 2, …, $N$ instead of 0, 1, …, $N$. Moreover, in many cases it will be more natural to work with the reversed sequence $B_N$, $B_{N-1}$, …, $B_1$ instead of $B_1$, $B_2$, …, $B_N$ (then the lower index indicates the number of steps up to the termination of the process). However, the main idea remains essentially unchanged.

Instead of exploring further the abstract description, we continue with the analysis of several examples which will serve as an illustration of the above concepts.

**1.1. A warm up.** Consider an investor, whose capital at $n$-th day is equal to $x_n$, $n = 0, 1, 2, \ldots, N$. At $m$-th day, the investor consumes $u_m$ of the capital and invests the remaining part; the interest rate is equal to $\gamma > 1$. That is, the sequence $(x_n)_{n=0}^N$ is governed by the equation

$$x_m = \gamma(x_{m-1} - u_m), \qquad m = 1, 2, \ldots, N,$$

where $u_m \leq x_{m-1}$ for each $m$. Suppose that the purpose of the investor is to maximize the functional $J(x_0, u_1, u_2, \ldots, u_N) = \sum_{m=1}^N u_m$. This can be easily solved by the above approach. Introduce the Bellman sequence

$$B_n(x) = \sup \left\{ \sum_{m=n+1}^{N} u_m \, : \, x_n = x \right\}.$$

We have $B_N(x) = 0$ and the Bellman equation gives

$$B_{N-1}(x) = \sup_{u \leq x} \left\{ u + B_N(\gamma(x-u)) \right\} = \sup_{u \leq x} u = x.$$

(This is perfectly intuitive: "there is no tomorrow", so the investor spends all the money). Furthermore,

$$B_{N-2}(x) = \sup_{u \leq x} \left\{ u + B_{N-1}(\gamma(x-u)) \right\} = \sup_{u \leq x} \left\{ u + \gamma(x-u) \right\} = \gamma x$$

(with supremum attained at zero),

$$B_{N-3}(x) = \sup_{u \leq x} \left\{ u + B_{N-2}(\gamma(x-u)) \right\} = \sup_{u \leq x} \left\{ u + \gamma^2(x-u) \right\} = \gamma^2 x$$

(with supremum attained at zero), etc.: for any $n = 0, 1, 2, \ldots, N-1$ we have $B_n(x) = \gamma^{N-n-1} x$ and the appropriate control is equal to zero. This implies that the investor should save the money for the last day, and then spend all the capital. This answer is obvious: keeping the capital unchanged maximizes the profit obtained via the interest rate.

**1.2. Towards analytic applications.** For a fixed $N$, we will compute the quantity

$$\sup \left\{ \frac{a_1}{a_0 + a_1} + \frac{a_2}{a_1 + a_2} + \ldots + \frac{a_N}{a_{N-1} + a_N} \right\},$$

where the supremum is taken over all positive numbers $a_0, a_1, a_2, \ldots, a_N$. This problem can be studied with the use of optimal control: consider the strategy $u_1$, $u_2, \ldots, u_N$ given by $u_n = a_n$. Then the sequence $x_0, x_1, x_2, \ldots, x_N$ given by $x_n = a_n$ consists of positive numbers and satisfies (1.1) with $f_n(x,u) = u$. Our goal is to maximize the functional

$$J^N(x_0, u_1, u_2, \ldots, u_N) = \sum_{m=1}^{N} \frac{u_m}{x_{m-1} + u_m},$$

which is of the form (1.2), with $g_m(x,u) = u/(x+u)$ and $r(x) = 0$. The Bellman sequence is given by

$$B_n(x) = \sup \left\{ \frac{a_{n+1}}{x_n + a_{n+1}} + \frac{a_{n+2}}{a_{n+1} + a_{n+2}} + \ldots + \frac{a_N}{a_{N-1} + a_N} \,\Big|\, x_n = x \right\},$$

where $x > 0$ and the supremum is taken over all positive numbers $a_{n+1}$, $a_{n+2}$, $\ldots$, $a_N$. By the Bellman equation, we see that for any $x > 0$ we have $B_N(x) = 0$,

$$\begin{aligned} B_{N-1}(x) &= \sup_{u>0} \{g_N(x,u) + B_N(f_N(x,u))\} \\ &= \sup_{u>0} \left\{ \frac{u}{x+u} + B_N(u) \right\} = \sup_{u>0} \frac{u}{x+u} = 1, \end{aligned}$$

$$B_{N-2}(x) = \sup_{u>0} \left\{ \frac{u}{x+u} + B_{N-1}(u) \right\} = \sup_{u>0} \left\{ \frac{u}{x+u} + 1 \right\} = 2,$$

and so on,

$$B_0(x) = \sup_{u>0} \left\{ \frac{u}{x+u} + B_1(u) \right\} = \sup_{u>0} \left\{ \frac{u}{x+u} + N - 1 \right\} = N.$$

This gives the answer to the problem: the supremum is equal to $N$. Note that the optimal controls do not exist.

**1.3. A probabilistic inequality.** Next, we will prove that for any $N \geq 1$ and any numbers $a_1$, $a_2$, $\ldots$, $a_N \in [0,1]$ we have

$$(1-a_1)(1-a_2)\ldots(1-a_N) \geq 1 - a_1 - a_2 - \ldots - a_N.$$

The first step is to rewrite the inequality in the form

$$-a_1 - a_2 - \ldots - a_N - (1-a_1)(1-a_2)\ldots(1-a_N) \leq -1.$$

This fits perfectly into the above scheme. We consider the strategy $(u_1, u_2, \ldots, u_N) = (a_1, a_2, \ldots, a_N)$ and define the sequence $x_0$, $x_1$, $x_2$, $\ldots$, $x_N$ by $x_0 = 1$ and

$$x_n = x_{n-1} \cdot (1 - a_n).$$

Then the sequence takes values in $[0,1]$ and satisfies the evolution equation (1.1) with $f_n(x,u) = x(1-u)$. We need to maximize the functional

$$J(x_0, u_1, u_2, \ldots, u_N) = \sum_{m=1}^{N} (-u_m) - x_N,$$

which is of the form (1.2) with $g_m(x,u) = -u$ and $r(x) = -x$. By the above discussion, we introduce the Bellman sequence by

$$B_n(x) = \sup \left\{ \sum_{m=n+1}^{N} (-u_m) - x_N \,\middle|\, x_n = x \right\}, \qquad x \in [0,1],$$

where the supremum is taken over all $u_{n+1}$, $u_{n+2}$, $\ldots$, $u_N \in [0,1]$. By the very definition, we have $B_N(x) = -x$ and, by the Bellman equation,

$$B_{N-1}(x) = \sup_{u\in[0,1]} \Big( -u + B_N(x(1-u)) \Big) = \sup_{u\in[0,1]} \big( u(x-1) - x \big) = -x,$$

where the supremum is attained for $u = 0$. The remaining functions $B_{N-2}$, $B_{N-3}$, $\ldots$, $B_0$ are computed identically: we have $B_0(x) = B_1(x) = \ldots = B_N(x) = -x$ for all $n$ and therefore

$$J(x_0, u_1, u_2, \ldots, u_N) \leq B_0(x_0) = -1,$$

which is the claim. Note that equality holds if and only if all the controls are zero: $a_1 = a_2 = \ldots = a_N = 0$.

**1.4. Modified approach: AM-GM inequality.** Now we will show how dynamic programming yields one of the most fundamental inequalities.

THEOREM 1.2. *For any positive integer $N$ and any nonnegative numbers $a_1$, $a_2$, …, $a_N$ we have*

$$\frac{a_1 + a_2 + \ldots + a_N}{N} \geq (a_1 a_2 \ldots a_N)^{1/N}.$$

*Equality holds if and only if $a_1 = a_2 = \ldots = a_N$.*

PROOF. Although the inequality fits into the scheme developed above (see Remark 1.3 below), it is instructive to discuss a slightly different approach. As previously, we start with rephrasing the desired claim into the problem of the optimization of some value function. This can be done as follows. Fix a nonnegative number $x$ and consider the quantity

$$\sup\Big\{a_1 a_2 \ldots a_N \,:\, a_1,\, a_2,\, \ldots,\, a_N \geq 0,\; a_1 + a_2 + \ldots + a_N = x\Big\}.$$

We need to show that for any $x$, the above supremum does not exceed $(x/N)^N$. Note that this problem is not of the form discussed at the beginning, since the functional does not have the additive form. However, the general methodology of "decomposing" the problem into simpler, similar sub-problems, which are then connected via induction argument, applies. Again, the controls are the numbers $a_1$, $a_2$, …, $a_N$. The sequence $x_0$, $x_1$, $x_2$, …, $x_N$ is given by $x_0 = x$ and

$$x_n = x_{n-1} - a_n, \qquad n = 1,\, 2,\, \ldots,\, N,$$

so that $x_n = a_{n+1} + a_{n+2} + \ldots + a_N$. The Bellman sequence is given by

$$B_n(x) = \sup\Big\{a_{n+1} a_{n+2} \ldots a_N \,\Big|\, x_n = x\Big\}, \qquad n = 0,\, 1,\, 2,\, \ldots,\, N-1.$$

The dynamic approach rests on writing the system of equations which govern the evolution of the Bellman sequence. By the very definition, we have $B_{N-1}(x) = x$, and the version of Bellman equation is the following: for any $n = 0, 1, \ldots, N-2$ and any $x \geq 0$ we have

$$B_n(x) = \sup_{t \in [0,x]} \big\{B_{n+1}(x-t) \cdot t\big\}. \tag{1.4}$$

Indeed, if $a_{n+1}$, $a_{n+2}$, …, $a_N$ are arbitrary nonnegative numbers summing up to $x$ and we denote $t = a_{n+1} \in [0, x]$, then

$$a_{n+1} a_{n+2} \ldots a_N = t \cdot a_{n+2} a_{n+3} \ldots a_N \leq B_{n+1}(x-t) \cdot t,$$

by the definition of $B_{n+1}$ and the fact that $a_{n+2} + a_{n+3} + \ldots + a_N = x - t$. Taking the supremum over all $a_{n+1}$, $a_{n+2}$, …, $a_N$ as above, gives the inequality "$\leq$" in (1.4). To get the reverse, we fix $t \in [0, x]$ and nonnegative numbers $a_{n+2}$, $a_{n+3}$, …, $a_N$ summing up to $x - t$. Then $t + a_{n+2} + a_{n+3} + \ldots + a_N = x$, so the definition of $B_n$ yields

$$t \cdot a_{n+2} a_{n+3} \ldots a_n \leq B_n(x).$$

Now, taking the supremum over $a_{n+2}$, $a_{n+3}$, …, $a_N$ as above gives $tB_{n+1}(x-t) \leq B_n(x)$, and since $t \in [0, x]$ was arbitrary, the identity (1.4) follows.

It remains to solve the recurrence. In general, this might be quite involved, but here the *conjecture* for the formula for $B_n$ is directly encoded in the problem:

$$B_n(x) = (x/(N-n))^{N-n}. \tag{1.5}$$

This conjecture is easily confirmed by induction. Let us briefly present the calculations, as they will be useful in the identification of the optimal controls. For $n = N - 1$ the hypothesis is true. Assuming the validity for a fixed $n + 1 \in \{2, 3, \ldots, N - 1\}$, we derive that the expression

$$B_{n+1}(x-t) \cdot t = \left(\frac{x-t}{N-n-1}\right)^{N-n-1} \cdot t,$$

considered as a function of $t$, attains its maximum for $t = x/(N-n)$ (only). Furthermore, this maximal value is equal to $(x/(N-n))^{N-n}$. This yields (1.5) and the claim follows.

The above calculations encode, for any fixed $x \geq 0$, the optimal controls $a_1^*$, $a_2^*$, ..., $a_N^*$ for $B_0(x)$. To see this, assume that $x > 0$ (for $x = 0$ there is nothing to prove). We go back to the above proof of (1.4). We have

$$B_0(x) = \sup_{t\in[0,x]} \{B_1(x-t) \cdot t\},$$

and the supremum is attained for the unique choice $t = x/N$. This necessarily implies that $a_1^*$ must be equal to $x/N$ and $a_2^* + a_3^* + \ldots + a_N^* = (N-1)x/N$. To get $a_2^*$, we make use of the following version of the optimality principle. We have

$$a_1^* a_2^* \ldots a_N^* = B_1(x) = B_2\left(x - \frac{x}{N}\right) \cdot \frac{x}{N} = B_2\left(x - \frac{x}{N}\right) \cdot a_1^*,$$

or, equivalently (recall that we have assumed $x > 0$)

$$a_2^* a_3^* \ldots a_N^* = B_2\left(x - \frac{x}{N}\right).$$

This brings us to the same position as above, with the length of the unknown extremal sequence decreased by 1 (and the required sum $x$ replaced by $x - x/N$). Repeating the arguments, we show that $a_2^* = (x - x/N)/(N-1) = x/N$, $a_3^* + a_4^* + \ldots + a_N = x - 2x/N$, and the numbers $a_3^*$, $a_4^*$, ..., $a_N^*$ satisfy

$$a_3^* a_4^* \ldots a_N^* = B_3\left(x - \frac{2x}{N}\right),$$

and so on. The procedure can be carried out until we get all the values of $a_1^*$, $a_2^*$, ..., $a_N^*$: one easily checks by induction that $a_1^* = a_2^* = \ldots = a_N^* = x/N$ is the extremal sequence we have searched for. □

Remark 1.3. There is an alternative approach to the AM-GM estimate. Consider the sequence $x_0 = 1$, $x_1 = a_1$ and

$$x_n = \frac{n^n}{(n-1)^{n-1}} \cdot x_{n-1} \cdot a_n, \qquad n = 1, 2, \ldots, N.$$

Then we have $x_n = n^n a_1 a_2 \ldots a_n$ and the inequality is equivalent to showing that the functional

$$-a_1 - a_2 - \ldots - a_N + x_N^{1/N}$$

is nonpositive. This fits into the scheme developed above.

**1.5. A higher-dimensional DP problem.** We turn to the case in which Bellman functions depend on more than one variable.

THEOREM 1.4. *For any nonnegative numbers* $a_1, a_2, \ldots, a_N, b_1, b_2, \ldots, b_N$ *we have*

$$(a_1+b_1)(a_2+b_2)\ldots(a_N+b_N) \geq ((a_1a_2\ldots a_N)^{1/N} + (b_1b_2\ldots b_N)^{1/N})^N. \tag{1.6}$$

PROOF. We may assume that all numbers $a_i$ and $b_j$ are non-zero (otherwise, the claim is obvious). For any $x$, $y > 0$, consider the function

$$B_n(x,y) = \inf \Big\{ (a_1+b_1)(a_2+b_2)\ldots(a_n+b_n) \Big\},$$

where the infimum is taken over all sequences $a_1, a_2, \ldots, a_n, b_1, b_2, \ldots, b_n$ of positive numbers such that $a_1a_2\ldots a_n = x$ and $b_1b_2\ldots b_n = y$. Clearly, we have $B_1(x,y) = x+y$, and Bellman equation becomes

$$B_{n+1}(x,y) = \inf \Big\{ (s+t)B_n(x/s, y/t) \Big\}, \tag{1.7}$$

where the infimum is taken over all $s$, $t > 0$. One easily shows by induction that the solution to this recurrence is given by $B_n(x,y) = (x^{1/n} + y^{1/n})^n$, and the infimum in (1.7) is attained for $s$, $t$ such that $y/x = (t/s)^{n+1}$. This establishes the desired inequality; furthermore, we obtain that the optimal controls $a_1^*, a_2^*, \ldots, a_N^*, b_1^*, b_2^*, \ldots, b_N^*$ satisfy

$$\frac{a_1^*a_2^*\ldots a_{n+1}^*}{b_1^*b_2^*\ldots b_{n+1}^*} = \left(\frac{a_{n+1}^*}{b_{n+1}^*}\right)^{1/(n+1)}, \qquad n = 1, 2, \ldots, N-1,$$

i.e., the equality in (1.6) is attained if and only if $a_1/b_1 = a_2/b_2 = \ldots = a_N/b_N$. □

## 2. Discrete system, infinite horizon

Now we turn our attention to the case in which we are interested in the control evolving in an infinite number of sets. There are essentially two types of approaches, it is best to present them on a concrete example.

THEOREM 1.5. *Prove that for any nonnegative numbers* $a_1, a_2, \ldots$ *we have*

$$\sum_{n=1}^{\infty} a_n \leq \sum_{n=1}^{\infty} 2^n a_n^2 + \frac{1}{4}. \tag{1.8}$$

PROOF, THE FIRST APPROACH. The idea is to make use of the method for the finite horizon, and then let the horizon go to infinity. More precisely, for any $N \geq 1$, we introduce the Bellman functions $B_N : \mathbb{R}_+ \to \mathbb{R}$ by

$$B_N(x) = \sup \left\{ \sum_{n=1}^{N} a_n \, : \, \sum_{n=1}^{N} 2^n a_n^2 = x \right\}.$$

We have $B_1(x) = \sqrt{\frac{x}{2}}$ and the Bellman equation reads

$$B_N(x) = \sup \left\{ u + B_{N-1}\left(x - 2^N u^2\right) \right\},$$

where the supremum is taken over all $u \geq 0$ such that $2^N u^2 \leq x$. This recurrence is not difficult to solve: after some straightforward calculations we compute that

$$B_N(x) = \sqrt{\frac{(2^N-1)x}{2^N}}.$$

It remains to perform a limiting argument: for any sequence $(a_n)_{n\geq 1}$ as in the statement and any $N$, we have

$$a_1 + a_2 + \ldots + a_N \leq B_N \left( \sum_{n=1}^{N} 2^n a_n^2 \right)$$
$$\leq \sqrt{\frac{2^N - 1}{2^N} \cdot \sum_{n=1}^{\infty} 2^n a_n^2} \leq \sqrt{\sum_{n=1}^{\infty} 2^n a_n^2} \leq \sum_{n=1}^{\infty} 2^n a_n^2 + \frac{1}{4}.$$

This gives the claim. □

However, sometimes such an approach might fail. It happens quite often that the Bellman sequence is extremely difficult (or impossible) to compute explicitly, however, its limit version can be handled efficiently. Let us see this alternative method.

Proof, the second approach. We introduce the *single* Bellman function $B : \mathbb{R}_+ \to \mathbb{R}$ given by

$$B(x) = \sup \left\{ \sum_{n=1}^{\infty} a_n \,:\, \sum_{n=1}^{\infty} 2^n a_n^2 = x \right\}. \tag{1.9}$$

Now the key is that the Bellman equation becomes a *functional equation* for $B$. Take any sequence $(a_n)_{n\geq 1}$ and note that

$$\sum_{n=1}^{\infty} a_n = a_1 + \sum_{n=2}^{\infty} a_n \qquad \text{and} \qquad \sum_{n=1}^{\infty} 2^n a_n^2 = 2a_1^2 + 2\sum_{n=2}^{\infty} 2^{n-1} a_n^2.$$

Therefore, we see that

$$B(x) = \sup_{u \in [0, \sqrt{x/2}]} \left\{ u + B\left( \frac{x - 2u^2}{2} \right) \right\}.$$

Now the analysis splits into two steps. First we need to solve the above equation, and then check rigorously that it does yield the desired estimate.

*I. Search for $B$.* Note that the above equation does not have unique solution: indeed, if $B$ satisfies it, then so does $B + c$ for any constant $c$. To find the right formula for $B$, or rather *guess it*, we look at the abstract definition. Note that the supremum in the definition of $B$ is taken over all sequences $(a_n)_{n\geq 1}$ with $\sum_{n=1}^{\infty} 2^n a_n^2 = x$. If we multiply $a_n$ by $\lambda$, then the latter sum multiplies by $\lambda^2$ (and hence it is equal to $\lambda^2 x$). On the other hand, then $\sum_{n=1}^{\infty} a_n$ multiplies by $\lambda$, and hence we must have $B(\lambda^2 x) = \lambda B(x)$. More formally,

$$B(\lambda^2 x) = \sup \left\{ \sum_{n=1}^{\infty} a_n \,:\, \sum_{n=1}^{\infty} 2^n a_n^2 = \lambda^2 x \right\} = \sup \left\{ \sum_{n=1}^{\infty} \lambda a_n' \,:\, \sum_{n=1}^{\infty} 2^n (a_n')^2 = x \right\}$$
$$= \lambda \sup \left\{ \sum_{n=1}^{\infty} a_n' \,:\, \sum_{n=1}^{\infty} 2^n (a_n')^2 = x \right\} = \lambda B(x),$$

where we have substituted $a_n = \lambda a'_n$. This implies that $B$ should be a root function: $B(x) = c\sqrt{x}$ for some constant $c > 0$. Let us go back to the Bellman equation:

$$c\sqrt{x} = \sup_u \left\{ u + c\sqrt{\frac{x - 2u^2}{2}} \right\}.$$

We easily check that the supremum is attained for $u = \sqrt{x/2(c^2+1)}$ and equals $\sqrt{(c^2+1)x/2}$. Thus the Bellman equation gives $c = \sqrt{\frac{c^2+1}{2}}$, or $c = 1$, and hence we have obtained the candidate for the Bellman function: $B(x) = \sqrt{x}$.

*II. Verification.* From the above construction, we know that $B(x) = \sqrt{x}$ satisfies the Bellman equation. Therefore, for any sequence $(a_n)_{n\geq 1}$,

$$\begin{aligned} B\left(\sum_{n=1}^{\infty} 2^n a_n^2\right) &= B\left(2a_1^2 + 2\sum_{n=2}^{\infty} 2^{n-1} a_n^2\right) \\ &\geq a_1 + B\left(\sum_{n=2}^{\infty} 2^{n-1} a_n^2\right) = a_1 + B\left(\sum_{n=1}^{\infty} 2^n a_{n+1}^2\right). \end{aligned} \tag{1.10}$$

Replacing $(a_n)_{n\geq 1}$ with $(a_{n+1})_{n\geq 1}$, we get

$$B\left(\sum_{n=1}^{\infty} 2^n a_{n+1}^2\right) \geq a_2 + B\left(\sum_{n=1}^{\infty} 2^n a_{n+2}^2\right) \tag{1.11}$$

and so on: by induction,

$$B\left(\sum_{n=1}^{\infty} 2^n a_n^2\right) \geq a_1 + a_2 + \ldots + a_N + B\left(\sum_{n=1}^{\infty} 2^n a_{n+N}^2\right) \geq a_1 + a_2 + \ldots + a_N.$$

This gives

$$a_1 + a_2 + \ldots + a_N \leq \sqrt{\sum_{n=1}^{\infty} 2^n a_n^2} \leq \sum_{n=1}^{\infty} 2^n a_n^2 + \frac{1}{4}, \tag{1.12}$$

and the assertion follows, since $N$ was arbitrary.

The above approach also easily gives that the constant $1/4$ is optimal. First, the last bound in (1.12) becomes equality if $\sum_{n=1}^{\infty} 2^n a_n^2 = 1/4$. Next, as we have proved above, for any $x$ the supremum

$$\sup_{u\in[0,\sqrt{x/2}]} \left\{ u + B\left(\frac{x - 2u^2}{2}\right) \right\}$$

is attained for $u = \sqrt{x}/2$. Consequently, equality holds in (1.10) for the choice $a_1 = \sqrt{1/4}/2 = 1/4$. Then $\sum_{n=2}^{\infty} 2^n a_n^2 = 1/8$ and hence $\sum_{n=2}^{\infty} 2^{n-1} a_n^2 = 1/16$. Thus the equality in (1.11) holds for $a_2 = \sqrt{1/16}/2 = 1/8$, and so on: we must take $a_n = 2^{-n-1}$ and then equality holds in (1.8). □

An important reformulation of the above solution is in order.

PROPOSITION 1.6. Fix $c \geq 0$ and consider the inequality

$$\sum_{n=1}^{\infty} a_n \leq \sum_{n=1}^{\infty} 2^n a_n^2 + c. \tag{1.13}$$

The estimate is valid, if there exists a function $B$ on $[0, \infty)$ satisfying

1° $B(x) \geq 0$;
2° $B(x) \leq x + c$;
3° For any $x$ and any $u \in [0, \sqrt{x/2}]$ we have

$$B(x) \geq u + B\left(\frac{x - 2u^2}{2}\right).$$

PROOF. Suppose that $B$ satisfies 1°, 2° and 3°. Exploiting the third condition, we may write

$$\begin{aligned} B\left(\sum_{n=1}^{\infty} 2^n a_n^2\right) &\geq a_1 + B\left(\sum_{n=1}^{\infty} 2^n a_{n+1}^2\right) \\ &\geq a_1 + a_2 + B\left(\sum_{n=1}^{\infty} 2^n a_{n+2}^2\right) \\ &\dots \\ &\geq a_1 + a_2 + \ldots + a_N + B\left(\sum_{n=1}^{\infty} 2^n a_{n+N}^2\right). \end{aligned}$$

Thus, by 1° and 2°,

$$\sum_{n=1}^{\infty} 2^n a_n^2 + c \geq a_1 + a_2 + \ldots + a_N.$$

It suffices to let $N \to \infty$ to get (1.13).

Actually, the reverse implication of the proposition is also true: we consider the abstract Bellman function (1.9). As shown above, it satisfies 1° and 3°, and the majorization 2° follows from the assumed validity of (1.13). □

We conclude with another instructive example.

THEOREM 1.7. *For any sequence $a_1$, $a_2$, …, $a_n$ of numbers belonging to $[0,1]$, we have*

$$\prod_{k=1}^{n}(1 - a_k) \leq 1 - \sum_{k=1}^{n} a_k + \frac{1}{2}\left(\sum_{k=1}^{n} a_k\right)^2. \tag{1.14}$$

PROOF. This inequality can be proven with the use of a Bellman function of one variable, but we will proceed differently and study the function of two variables.

We rewrite the inequality in the form

$$\frac{\prod_{k=1}^{n}(1 - a_k)}{1 - \sum_{k=1}^{n} a_k + \frac{1}{2}\left(\sum_{k=1}^{n} a_k\right)^2} \leq 1$$

and write down the associated Bellman function

$$B(x,y) = \sup\left\{\frac{x\prod_{k=1}^{n}(1 - a_k)}{1 - (y + \sum_{k=1}^{n} a_k) + \frac{1}{2}\left(y + \sum_{k=1}^{n} a_k\right)^2}\right\},$$

the supremum taken over all $n$. We want to show that $B(1,0) \leq 1$.

The next step is to write the Bellman equation. We see that picking $a_1$ makes $x$ go to $x(1 - a_1)$ and $y$ go to $y + a_1$: hence

$$B(x,y) = \sup_{a>0} B(x(1-a), y + a).$$

Now, as previously, we need to find the solution of this equation, and we start with guessing. By the very definition of $B$, we have $B(x,y) = xB(1,y) =: x\varphi(y)$. Furthermore, if we plug $a = 0$, the supremum is attained: therefore,

$$-xB_x(x,y) + B_y(x,y) = \frac{d}{da}B(x(1-a), y+a)\Big|_{a=0} \leq 0.$$

In the language of $\varphi$, this means $-x\varphi(y) + x\varphi'(y) \leq 0$, or $\varphi'(y) \leq \varphi(y)$. Assume equality: we obtain $\varphi(y) = Ke^y$. Thus we have constructed the candidate $B(x,y) = Kxe^y$. It is easy to check that it satisfies the Bellman equation: $xe^y \geq x(1-a)e^{y+a}$ is equivalent to $e^{-a} \geq 1-a$. Now, if we put $K = 1$, then $B(1,0) = 1$; furthermore, we have $B(x,y) \geq x/(1-y+y^2/2)$ (a simple verification). Therefore,

$$\begin{aligned} 1 = B(1,0) &\geq B(1-a_1, a_1) \geq B((1-a_1)(1-a_2), a_1+a_2) \geq \dots \\ &\geq B((1-a_1)\dots(1-a_N), a_1+\dots+a_N) \geq \frac{\prod_{k=1}^n (1-a_k)}{1-\sum_{k=1}^n a_k + \frac{1}{2}\left(\sum_{k=1}^n a_k\right)^2}. \end{aligned}$$

This gives the desired claim. □

Here is the analogue of Proposition 1.6.

Remark 1.8. The estimate (1.6) holds true if and only if there is a function $B$ on $[0,\infty)^2$ satisfying

1° $B(1,0) \leq 1$;

2° $B(x,y) \geq \dfrac{x}{1-y+y^2/2}$ for all $x$, $y \geq 0$;

3° For all $x$, $y \geq 0$ and all $a \in [0,1]$, we have

$$B(x,y) \geq B(x(1-a), y+a).$$

## 3. Integral inequalities

**3.1. A warm up: Hardy inequality.** Let $1 < p < \infty$. We will show how the Bellman function approach can be used to obtain the estimate

$$\int_0^\infty \left|\frac{1}{t}\int_0^t f\right|^p \mathrm{d}t \leq C_p^p \int_0^\infty |f|^p, \tag{1.15}$$

with the optimal choice $C_p = p/(p-1)$.

We start by noting that $C_p$ must be at least 1 (plug the constant $f$). Next, it is enough to show the bound for nonnegative $f$. Furthermore, it suffices to show that for any $T > 0$,

$$\int_0^T \left[\left(\frac{1}{s}\int_0^s f\right)^p - C_p^p f^p(s)\right] \mathrm{d}s \leq 0. \tag{1.16}$$

We fix $T$ and introduce the abstract Bellman function $\mathbb{B} : [0,T] \times [0,\infty) \to \mathbb{R}$ by

$$\mathbb{B}(t,x) = \sup\left\{\int_0^t \left(\frac{1}{s}\int_0^s f\right)^p - C_p^p f^p(s)\,\mathrm{d}s\right\},$$

where the supremum is taken over all piecewise continuous $f : [0,t] \to [0,\infty)$ satisfying $\frac{1}{t}\int_0^t f = x$.

Lemma 1.9. *If the inequality* (1.16) *holds true, then the function* $\mathbb{B}$ *satisfies the conditions*

*1° We have* $\mathbb{B}(0,x) = 0$ *for all* $x \geq 0$.

*2° For any $x \geq 0$ we have $\mathbb{B}(T, x) \leq 0$.*

PROOF. Perhaps it would me more instructive to express 1° as $\mathbb{B}(0, x) \geq 0$ (but this does not make special sense in the light of 2°). The piecewise continuity guarantees that the limits of $\frac{1}{s}\int_0^s f$ and $f(s)$ as $s \to 0$ exist, and hence any integral in the definition of $\mathbb{B}(0, x)$ vanishes. The majorization 2° is a direct consequence of (1.16). □

Now we will provide an appropriate form of the Bellman equation (or rather, Bellman inequality).

LEMMA 1.10. *3° Assume that $\mathbb{B}$ is of class $C^1$. Then we have*

$$\inf_u \left\{ \mathbb{B}_t(t, x) + \mathbb{B}_x(t, x) \cdot \frac{u - x}{t} - x^p + C_p^p u^p \right\} \geq 0 \tag{1.17}$$

*for any $(t, x) \in (0, T] \times \mathfrak{D}$.*

PROOF. Fix $t > 0$, $x \geq 0$ and a small auxiliary parameter $\varepsilon \in (0, t)$. Take an arbitrary function $f$ as in the definition of $B(t, x)$ and extend it to a function $\tilde{f}$ on $[0, t + \varepsilon]$, setting $f \equiv u \geq 0$ on $(t, t + \varepsilon]$. Then for $s \geq t$,

$$\frac{1}{t+\varepsilon} \int_0^{t+\varepsilon} \tilde{f} = \frac{1}{t+\varepsilon} \left( \int_0^t f + \int_t^{t+\varepsilon} u \right) = \frac{tx + \varepsilon u}{t + \varepsilon}$$

and consequently, by the very definition of the Bellman function,

$$\begin{aligned} \mathbb{B}\left(t+\varepsilon, \frac{tx+\varepsilon u}{t+\varepsilon}\right) &\geq \int_0^{t+\varepsilon} \left(\frac{1}{s}\int_0^s \tilde{f}\right)^p - C_p^p \tilde{f}^p(s)\,\mathrm{d}s \\ &= \int_0^t \left(\frac{1}{s}\int_0^s f\right)^p - C_p^p f^p(s)\,\mathrm{d}s \\ &\quad + \int_t^{t+\varepsilon} \left(\frac{tx + (s-t)u}{s}\right)^p - C_p^p u^p\,\mathrm{d}s. \end{aligned}$$

But $f$ was chosen arbitrarily; therefore

$$\begin{aligned} \mathbb{B}\left(t+\varepsilon, \frac{tx+\varepsilon u}{t+\varepsilon}\right) &\geq \mathbb{B}(t, x) + \int_t^{t+\varepsilon} \left(\frac{tx + (s-t)u}{s}\right)^p - C_p^p u^p\,\mathrm{d}s \\ &= \mathbb{B}(t, x) + \varepsilon(x^p - C_p^p u^p) + o(\varepsilon). \end{aligned}$$

Now we exploit the assumed smoothness of $\mathbb{B}$. We put all the terms on the left, divide both sides by $\varepsilon$ and let $\varepsilon \to 0$. As the result, we obtain

$$\mathbb{B}_t(t, x) + \mathbb{B}_x(t, x) \cdot \frac{u - x}{t} - x^p + C_p^p u^p \geq 0. \tag{1.18}$$

The number $u$ was chosen arbitrarily: this yields the desired Bellman inequality:

$$\inf_u \left\{ \mathbb{B}_t(t, x) + \mathbb{B}_x(t, x) \cdot \frac{u - x}{t} - x^p + C_p^p u^p \right\} \geq 0.$$

□

The beautiful feature of the approach is that the implication can be reversed.

LEMMA 1.11. *Suppose that* some function *$B$ is of class $C^1$ and satisfies 1°, 2° and 3°. Then* (1.15) *holds true.*

PROOF. We may assume that $f$ is nonnegative and piecewise continuous. Observe that by (1.17), we have

$$\begin{aligned}\frac{d}{dt}B\left(t,\frac{1}{t}\int_0^t f\right) &= B_t\left(t,\frac{1}{t}\int_0^t f\right)+B_x\left(t,\frac{1}{t}\int_0^t f\right)\cdot\frac{f(t)-\frac{1}{t}\int_0^t f}{t}\\ &\geq \left(\frac{1}{t}\int_0^t f\right)^p - C_p^p f(t)^p.\end{aligned}$$

Consequently, integrating, we get

$$B\left(T,\frac{1}{T}\int_0^T f\right)-B(0,f(0))\geq \int_0^T\left(\frac{1}{t}\int_0^t f\right)^p - C_p^p f(t)^p \mathrm{d}t.$$

It remains to note that $-B(0,f(0))\leq 0$ (by 1°) and $B\left(T,\frac{1}{T}\int_0^T f\right)\leq 0$ (by 2°). □

So, as in the discrete context, we see that the validity of the estimate (1.15) is (almost) equivalent to the existence of a function $B$ satisfying 1°, 2° and 3°. The only problem is that 3° involves differentiation, which might not be allowed. There is an alternative condition, which makes the function work in general:

3°' For any piecewise continuous $f:[0,T]\to\mathbb{R}$, the function

$$t\mapsto B\left(t,\frac{1}{t}\int_0^t f\right)+\int_t^T\left[\left(\frac{1}{s}\int_0^s f\right)^p - C_p^p f(s)^p\right]\mathrm{d}s$$

is nondecreasing.

Now we apply the usual procedure.

*On the search for the Bellman function.* Let us first extract some homogeneity-type properties of $B$.

LEMMA 1.12. *We have*

$$\mathbb{B}(t,\lambda x)=\lambda^p B(t,x)$$

*and*

$$\mathbb{B}(t,x)=t\mathbb{B}(1,x).$$

PROOF. Observe that $f$ is as in the definition of $\mathbb{B}(t,x)$ if and only if $\lambda f$ is as in the definition of $\mathbb{B}(t,\lambda x)$. Therefore,

$$\begin{aligned}\mathbb{B}(t,\lambda x) &= \sup\left\{\int_0^t\left(\frac{1}{s}\int_0^s f(u)\,\mathrm{d}u\right)^p - C_p^p f^p(s)\,\mathrm{d}s\,:\,\frac{1}{t}\int_0^t f=\lambda x\right\}\\ &= \sup\left\{\int_0^t\left(\frac{1}{s}\int_0^s \lambda f(u)\,\mathrm{d}u\right)^p - C_p^p(\lambda f)^p(s)\,\mathrm{d}s\,:\,\frac{1}{t}\int_0^t \lambda f=\lambda x\right\}\\ &= \lambda^p\sup\left\{\int_0^t\left(\frac{1}{s}\int_0^s f(u)\,\mathrm{d}u\right)^p - C_p^p f^p(s)\,\mathrm{d}s\,:\,\frac{1}{t}\int_0^t f=x\right\}\\ &= \lambda^p\mathbb{B}(t,x).\end{aligned}$$

To prove the second homogeneity, pick an arbitrary $f$ as in the definition of $\mathbb{B}(t,x)$ and consider its dilation $f^{(t)}:[0,1]\to[0,\infty)$ given by $f^{(t)}(u)=f(tu)$. We have

$$\int_0^t\left(\frac{1}{s}\int_0^s f(u)\,\mathrm{d}u\right)^p - c_p f^p(s)\,\mathrm{d}s = t\int_0^1\left(\frac{1}{s}\int_0^s f^{(t)}(u)\mathrm{d}u\right)^p - c_p(f^{(t)}(s))^p\mathrm{d}s$$

and if $f$ runs over the class of all nonnegative functions on $(0,t]$ satisfying the condition $\frac{1}{t}\int_0^t f(u)\mathrm{d}u = x$, then $f^{(t)}$ runs over the class of all nonnegative functions on $(0,1]$ of average $x$. Thus, taking the supremum over $f$ and $f^{(t)}$, we get the desired homogeneity. □

By the above lemma, we immediately obtain

$$\mathbb{B}(t,x) = t\mathbb{B}(1,x) = tx^p\mathbb{B}(1,1) =: -atx^p, \tag{1.19}$$

with $-a = \mathbb{B}(1,1) \leq 0$ to be found (the value must be nonpositive, by 2°). In particular, we see that $\mathbb{B}$ is of class $C^\infty$

$$\inf_u \left\{ \mathbb{B}_T(T,x) + \mathbb{B}_x(T,x)\cdot\frac{u-x}{T} - x^p + C_p^p u^p \right\} \geq 0.$$

Now we assume equality and recall the homogeneity obtained in (1.19): the equation becomes

$$\inf_u \left\{ -ax^p - pax^{p-1}(u-x) - x^p + C_p^p u^p \right\} = 0$$

or if we divide both sides by $u^p$ and make the substitution $s := x/u \geq 0$,

$$\inf_s \left\{ -as^p - pas^{p-1}(1-s) - s^p + C_p^p \right\} = 0.$$

This can be rewritten as

$$C_p^p = \sup_s \left\{ s^p(1-(p-1)a) + pas^p \right\} = \frac{a^p(p-1)^{p-1}}{(a(p-1)-1)^{p-1}}.$$

Now we compute easily that the obtained constant $C_p$ is the smallest if $a = p/(p-1)$, and then $C_p^p = (p/(p-1))^p$. Thus we have obtained the value of the constant, and the associated Bellman function

$$B(x,y) = -\frac{p}{p-1}Tx^p.$$

*Verification.* The conditions 1°, 2° are trivial, the property 3° is guaranteed by the construction.

**3.2. General description.** Fix $d \geq 1$ and $T > 0$. Introduce a function $\Phi : (0,T]\times[0,\infty) \to \mathbb{R}^d$ and, for each $t \in (0,T]$, the auxiliary functional $X_t$ acting on nonnegative functions on $(0,t]$ by the formula

$$X_t(f) = \frac{1}{t}\int_0^t \Phi(u, f(u))\mathrm{d}u, \qquad t \in (0,T]. \tag{1.20}$$

Suppose that all functionals $X_t$ have equal range, denoted by $\mathfrak{D} \subseteq \mathbb{R}^d$. Next, let $F : (0,T]\times[0,\infty)\times\mathfrak{D} \to [0,\infty)$, $G : (0,T]\times\mathfrak{D} \to [0,\infty)$ be two given Borel functions. Suppose that we are interested in showing the estimate

$$\int_0^T F(t, f(t), X_t(f))\,\mathrm{d}t \leq G(T, X_T(f)) \tag{1.21}$$

for all $f$. For instance, fix $p \in (1,\infty)$ and let $\Phi(u,v) = (v, v^p)$; then

$$X_t(f) = \left(\frac{1}{t}\int_0^t f(u)\mathrm{d}u,\ \frac{1}{t}\int_0^t f(u)^p\,\mathrm{d}u\right)$$

takes values in the set $\mathfrak{D} = \{(x, y) \in [0, \infty)^2 : x^p \le y\}$. Now, set $F(t, u, x_1, x_2) = t^\alpha x_1^q$ and $G(x, y) = c_{p,q} x_2^{q/p}$, where $\alpha$, $q$ and $c_{p,q}$ are some positive constants. Then (1.21) becomes the localized Bliss' inequality

$$\int_0^T t^\alpha \left( \frac{1}{t} \int_0^t f(u) \mathrm{d}u \right)^q \mathrm{d}t \le c_{p,q} \left( \int_0^T f^p(t) \mathrm{d}t \right)^{q/p}.$$

Here is the version of the Bellman function technique which can be used to study the above setup. The proof is exactly the same as in the previous subsection.

LEMMA 1.13. *Assume that $\Phi$ is a continuous function. Suppose there exists a function $B : (0, T] \times \mathfrak{D} \to \mathbb{R}$ of class $C^1$, satisfying*

*1° (Nonnegativity) For any $t \in (0, T]$ and $x \in \mathfrak{D}$, we have $B(t, x) \ge 0$.*

*2° (Majorization) For any $x \in \mathfrak{D}$ we have $B(1, x) \le G(x)$.*

*3° (Bellman inequality) and satisfies*

$$\inf_u \left\{ B_t(t, x) + \left\langle \nabla_x B(t, x), \frac{\Phi(t, u) - x}{t} \right\rangle - F(t, u, x) \right\} \ge 0$$

*for any $(t, x) \in (0, T] \times \mathfrak{D}$.*

*Then the inequality* (1.21) *holds.*

There is a weaker version 3°, which does not involve differentiability (replacing 3° with 3°' below, we may remove the assumption on continuity of $\Phi$ and differentiability of $B$).

3°' (Bellman monotonicity) For any nonnegative $f$ on $(0, T]$, the function

$$t \mapsto B(t, X_t(f)) + \int_t^1 F(u, f(u), X_u(f)) \mathrm{d}u$$

is nondecreasing on $(0, T]$.

We also have the implication in the reverse direction.

LEMMA 1.14. *The abstract Bellman function*

$$\mathbb{B}(t, x) = \sup \left\{ \int_0^t F(u, f(u), X_u(f)) \, du \right\},$$

*where the supremum is taken over all piecewise continuous $f$ satisfying $X_t(f) = x$, satisfies 1°, 2° and 3°'. If $\mathbb{B}$ is of class $C^1$, then it satisfies 3° as well.*

## 4. Problems

**1.** For any positive numbers $a_1$, $a_2$, ..., $a_n$ we have

$$(1 + a_1)(1 + a_2) \dots (1 + a_n) \ge (1 + (a_1 a_2 \dots a_n)^{1/n})^n.$$

**2.** Prove that for any nonnegative numbers $a_1$, $a_2$, ..., $a_N$ satisfying $x = a_1 + a_2 + \dots + a_N \le 1$ we have

$$\frac{a_1}{1 + a_1^2} + \frac{a_2}{1 + a_2^2} + \dots + \frac{a_N}{1 + a_N^2} \le \frac{N^2 x}{N^2 + x^2}.$$

**3.** Prove that for any $n \ge 1$ and any real numbers $a_1$, $a_2$, ..., $a_n$ we have

$$a_1 a_2 \dots a_n \le \frac{a_1^2}{2} + \frac{a_2^4}{4} + \frac{a_3^8}{8} + \dots + \frac{a_n^{2^n}}{2^n} + \frac{1}{2^n}.$$

**4.** Prove that for any positive numbers $a_1$, $a_2$, …, $a_n$, $b_1$, $b_2$, …, $b_n$ we have

$$\left(\frac{a_1+a_2+\ldots+a_n}{b_1+b_2+\ldots+b_n}\right)^2 \le \left(\frac{a_1}{b_1}\right)^2+\left(\frac{a_2}{b_2}\right)^2+\ldots+\left(\frac{a_n}{b_n}\right)^2.$$

**5.** For any positive integer $N$ and any nonnegative numbers $a_1$, $a_2$, …, $a_N$, $b_1$, $b_2$, …, $b_N$, we have

$$\sum_{n=1}^{N} n(b_n-1)(a_1a_2\ldots a_n)^{1/n}+N(a_1a_2\ldots a_N)^{1/N} \le \sum_{n=1}^{N} a_n b_n^n.$$

**6.** Prove that if $a_1 \le a_2 \le \ldots \le a_n$ and $b_1 \le b_2 \le \ldots \le b_n$, then

$$(a_1+a_2+\ldots+a_n)(b_1+b_2+\ldots+b_n) < n(a_1b_1+a_2b_2+\ldots+a_nb_n).$$

**7.** Prove that for any positive numbers $a_1$, $a_2$, …, $a_n$ satisfying $a_1+a_2+\ldots+a_n<1$ we have

$$\frac{a_1a_2\ldots a_n\big(1-(a_1+a_2+\ldots+a_n)\big)}{(a_1+a_2+\ldots+a_n)(1-a_1)(1-a_2)\ldots(1-a_n)} \le \frac{1}{n^{n+1}}.$$

**8.** Prove that for any positive numbers $a_1$, $a_2$, …, $a_n$ satisfying $a_1a_2\ldots a_n=1$ we have

$$\frac{1}{n-1+a_1}+\frac{1}{n-1+a_2}+\ldots+\frac{1}{n-1+a_n} \le 1.$$

**9.** Prove that for any positive numbers $a_1$, $a_2$, …, $a_n$ we have

$$\left(1+\frac{1}{a_1}\right)\left(1+\frac{1}{a_2}\right)\ldots\left(1+\frac{1}{a_n}\right) \ge \left(1+\frac{n}{a_1+a_2+\ldots+a_n}\right)^n.$$

**10.** Prove that for any positive numbers $a_1$, $a_2$, …, $a_n$ with $a_1+a_2+\ldots+a_n=x$ we have

$$1+x \le (1+a_1)(1+a_2)\ldots(1+a_n) \le e^x.$$

**11.** Prove that for any $N \ge 2$ and any numbers $a_1$, $a_2$, …, $a_N$ belonging to $[0,1]$ we have

$$\prod_{n=1}^{N}(1-a_n) \le 1-\sum_{n=1}^{N} a_n+\sum_{1\le n<m\le N} a_na_m.$$

**12.** Find the optimal constant $C$ in the inequality

$$\sum_{n=0}^{\infty}\frac{x_n^2}{x_{n+1}} \ge Cx_0,$$

to be valid for any nonincreasing sequence $(x_n)_{n\ge 0}$ of positive numbers.

**13.** Find the best constant $C$ in the inequality

$$\int_0^\infty \exp\left(\frac{1}{t}\int_0^t f\right) \le C\int_0^\infty \exp(f),$$

to be valid for all integrable $f$.

**14.** (F. Carlson [**4**]) Prove that there is a constant $C<\infty$ with the following property: for any $f:[0,\infty)\to\mathbb{R}$ we have

$$\int_0^\infty |f(t)|\mathrm{d}t \le C\left(\int_0^\infty f^2(u)\mathrm{d}u\right)^{1/4}\left(\int_0^\infty f^2(u)u^2\mathrm{d}u\right)^{1/4}.$$

CHAPTER 2

# Optimal stopping

## 1. Discrete time

**1.1. Martingale approach: description.** Suppose that $(\Omega, \mathcal{F}, \mathbb{P})$ is a given probability space, equipped with a filtration $(\mathcal{F}_n)_{n\geq 0}$, i.e., a nondecreasing family of sub-$\sigma$-algebras of $\mathcal{F}$. Let $G = (G_n)_{n\geq 0}$ be an adapted sequence of random variables, i.e., we assume that for each $n \geq 0$ the random variable $G_n$ is measurable with respect to $\mathcal{F}_n$. Our objective will be to stop this sequence so that the expected return is maximized. The stopping procedure is described by a random variable $\tau : \Omega \to \{0, 1, 2, \ldots\}$, which returns the value of the time when the sequence $(G_n)_{n\geq 0}$ should be stopped. Clearly, a reasonable procedure decides whether to stop the sequence at time $n$ or not based on the observations up to time $n$; this means that for each $n$ we have

$$\{\tau = n\} \in \mathcal{F}_n \qquad \text{for each } n \geq 0.$$

The random variable $\tau$ satisfying the above condition will be called a stopping time.

Let us put the discussion into a more precise framework. The optimal stopping problem concerns the study of

$$V_0 = \sup_{\tau} \mathbb{E} G_\tau, \tag{2.1}$$

where the supremum is taken over a certain family of adapted stopping times $\tau$ (which depends on the problem). We should point out that the study consists of two parts: (i) to compute the value $V_0$ as explicitly as possible; (ii) to identify the optimal stopping time $\tau_*$ (or the family of almost-optimal stopping times) for which the supremum $V_0$ is attained.

The first problem we encounter concerns the existence of $\mathbb{E}G_\tau$, to overcome which we need to impose some additional assumptions on $G$ and $\tau$. For example, if

$$\mathbb{E} \sup_{n\geq 0} |G_n| < \infty, \tag{2.2}$$

then the expectation $\mathbb{E}G_\tau$ is well defined for all stopping times $\tau$. Another possibility is to restrict in (2.1) to those $\tau$, for which the expectation exists. One way or another, we should emphasize that in general, this obstacle is just a technicality which is easily removed by some straightforward arguments (which might depend on the problem under the study). For the sake of simplicity and the clarity of the statements, we will assume that the condition (2.2) is satisfied, but it will be evident how to relax this requirement in other contexts.

So, let us assume that the supremum in (2.1) is taken over the family $\mathcal{M}$ of all stopping times $\tau$. A successful treatment of this problem requires the introduction, for each $n \leq N$, the smaller family

$$\mathcal{M}_n^N = \{\tau \in \mathcal{M} : n \leq \tau \leq N\}.$$

We will also write $\mathcal{M}^N = \mathcal{M}_0^N$ and $\mathcal{M}_n = \mathcal{M}_n^\infty$. These families give rise to the related value functions

$$V_n^N = \sup_{\tau \in \mathcal{M}_n^N} \mathbb{E} G_\tau, \tag{2.3}$$

and we will use the notation $V^N = V_0^N$, $V_n = V_n^\infty$ and $V = V_0^\infty$. The primary goal of this section is to present the solution to (2.3) with the use of martingale approach.

**1.2. Martingale approach: finite horizon.** If $N < \infty$ (the case of "finite horizon"), then the problem (2.3) can be easily solved by means of the backward induction. Indeed, let us fix a nonnegative integer $N$ and try to inspect the value functions as $n$ decreases from $N$ to 0. If $n = N$, then the class $\mathcal{M}_n^N$ consists of one stopping time $\tau \equiv N$ only and hence the optimal gain is equal to $G_N$ (and $V_N^N = \mathbb{E} G_N$). If $n = N - 1$, then we have two choices for the stopping time: we can either stop at time $N - 1$ or continue and stop at time $N$. In the first case our gain is $G_{N-1}$; in the second case we do not know what the random variable $G_N$ will be, so we can only say that on average, we will obtain $\mathbb{E}(G_N | \mathcal{F}_{N-1})$. Therefore, if $G_{N-1} \geq \mathbb{E}(G_N | \mathcal{F}_{N-1})$, we should stop immediately; otherwise, we should continue. For smaller values of $n$ we proceed similarly. More precisely, define recursively the sequence $(B_n^N)_{0 \leq n \leq N}$, representing the optimal gains at times 0, 1, 2, ..., $N$, as follows:

$$\begin{aligned} B_N^N &= G_N, \\ B_n^N &= \max\left\{G_n, \mathbb{E}(B_{n+1}^N | \mathcal{F}_n)\right\}, \qquad n = N-1,\, N-2,\, \ldots. \end{aligned} \tag{2.4}$$

The above discussion also suggests to consider the family of stopping times

$$\tau_n^N = \inf\left\{k \in \{n,\, n+1,\, \ldots,\, N\} \,:\, B_k^N = G_k\right\}, \tag{2.5}$$

for $n = 0,\, 1,\, 2,\, \ldots,\, N$.

THEOREM 2.1. *Suppose that $N$ is a fixed integer and the sequence $G = (G_k)_{k=n}^N$ satisfies $\mathbb{E} \max_{n \leq k \leq N} |G_k| < \infty$. Consider the optimal stopping problem* (2.3) *and the sequence $(B_k^N)_{k=n}^N$, defined by* (2.4).

*(i) The sequence $(B_k^N)_{k=n}^N$ is the smallest supermartingale majorizing $(G_k)_{k=n}^N$. In addition, the stopped sequence $\left(B_{\tau_n^N \wedge k}^N\right)_{k=n}^N$ is a martingale.*

*(ii) For any $0 \leq n \leq N$ we have, with probability 1,*

$$B_n^N \geq \mathbb{E}(G_\tau | \mathcal{F}_n) \qquad \textit{for any } \tau \in \mathcal{M}_n^N, \tag{2.6}$$

$$B_n^N = \mathbb{E}(G_{\tau_n^N} | \mathcal{F}_n). \tag{2.7}$$

*(iii) The stopping time $\tau_n^N$ is optimal in* (2.3) *and any other optimal stopping time $\tau_*$ satisfies $\tau_* \geq \tau_n^N$ almost surely.*

PROOF. (i) The supermartingale property and the majorization follow directly from the definition of the sequence $(B_n^N)_{n=0}^N$. If $(\bar{B}_n^N)_{n=0}^N$ is another supermartingale majorizing $(G_k)_{k=n}^N$, then the desired inequality $B_k^N \leq \bar{B}_k^N$ almost surely, $k = n,\, n+1,\, N+2,\, \ldots,\, N$, can be proved by backward induction. Indeed, the estimate is trivial for $k = N$ (we have $B_N^N = G_N \leq \bar{B}_N^N$, by the majorization property of $\bar{B}$), and assuming its validity for $k$, we see that

$$\bar{B}_{k-1}^N \geq \max\{G_{k-1}, \mathbb{E}(\bar{B}_k^N | \mathcal{F}_{k-1})\} \geq \max\{G_{k-1}, \mathbb{E}(B_k^N | \mathcal{F}_{k-1})\} = B_{k-1}^N.$$

So, it remains to prove the martingale property of the stopped process $\left(B^N_{\tau^N_n\wedge k}\right)^N_{k=n}$. We compute directly that

$$\begin{aligned}\mathbb{E}\left[B^N_{\tau^N_n\wedge(k+1)}\mid\mathcal{F}_k\right] &= \mathbb{E}\left[B^N_{\tau^N_n\wedge(k+1)}1_{\{\tau^N_n\le k\}}\mid\mathcal{F}_k\right]+\mathbb{E}\left[B^N_{\tau^N_n\wedge(k+1)}1_{\{\tau^N_n> k\}}\mid\mathcal{F}_k\right]\\ &= \mathbb{E}\left[B^N_{\tau^N_n\wedge k}1_{\{\tau^N_n\le k\}}\mid\mathcal{F}_k\right]+\mathbb{E}\left[B^N_{k+1}1_{\{\tau^N_n> k\}}\mid\mathcal{F}_k\right]\\ &= B^N_{\tau^N_n\wedge k}1_{\{\tau^N_n\le k\}}+1_{\{\tau^N_n> k\}}\mathbb{E}\left[B^N_{k+1}\mid\mathcal{F}_k\right].\end{aligned}$$

However, on the set $\{\tau^N_n>k\}$ we have $B^N_k>G_k$ and hence $B^N_k=\mathbb{E}(B^N_{k+1}|\mathcal{F}_k)$. This shows the identity $\mathbb{E}\left[B^N_{\tau^N_n\wedge(k+1)}\mid\mathcal{F}_k\right]=B^N_{\tau^N_n\wedge k}1_{\{\tau^N_n\le k\}}+1_{\{\tau^N_n>k\}}B^N_k=B^N_{\tau^N_n\wedge k}$ and part (i) is established.

(ii) This follows at once from (i) and Doob's optional sampling theorem.

(iii) Taking the expectations in (2.6) and (2.7) gives $\mathbb{E}G_\tau\le\mathbb{E}B^N_n=\mathbb{E}G_{\tau^N_n}$ for all $\tau\in\mathcal{M}^N_n$, showing that $\tau^N_n$ is indeed the optimal stopping time. Suppose that $\tau_*$ is another optimal stopping time. Then $B^N_{\tau_*}=G_{\tau_*}$ almost surely, since otherwise we would have

$$\mathbb{E}G_{\tau_*}<\mathbb{E}B^N_{\tau_*}\le\mathbb{E}B^N_n=\mathbb{E}G_{\tau^N_n},$$

where the second inequality follows from Doob's optional sampling theorem and the supermartingale property of the sequence $(B^N_k)^N_{k=n}$. The contradiction shows that $B_{\tau_*}$ and $G_{\tau_*}$ must coincide, and clearly $\tau^N_n$ is the smallest stopping time which has this property. □

REMARK 2.2. The following observation is an immediate consequence of the above considerations. Namely, as we have just proved, the stopping time $\tau^N_0$ is optimal for $V^N_0$. If it was not optimal to stop at the instances 0, 1, …, $n-1$, then the optimal policy coincides with that corresponding to the problem $V^N_n$. This naturally brings us into the context of dynamic programming studied in the previous chapter.

**1.3. Martingale approach: infinite horizon.** The above method required $N$ to be a finite integer, since we have needed the variable $G_N$ to start the backward recurrence. In the case $N=\infty$ one could try to use approximation-type arguments (of the form $V^\infty_n=\lim_{N\to\infty}V^N_n$), but these do not necessarily work in general, so we will proceed in a different manner. By (2.6) and (2.7) it seems tempting to write

$$B^N_n=\sup_{\tau\in\mathcal{M}^N_n}\mathbb{E}(G_\tau|\mathcal{F}_n).$$

However, two problems arise. The first is that (2.6) and (2.7) hold true on a set of full measure only which might depend on the stopping time, so the above identity might fail to hold. A second obstacle is that the supremum on the right need not be even measurable. To overcome these difficulties, a typical argument in the theory of optimal stopping is to introduce the concept of essential supremum.

DEFINITION 2.1. Let $(Z_\alpha)_{\alpha\in I}$ be a family of random variables. Then there is a countable subset $J$ of $I$ such that the random variable $\overline{Z}=\sup_{\alpha\in J}Z_\alpha$ satisfies the following two properties:

(i) $\mathbb{P}(Z_\alpha\le\overline{Z})=1$ for each $\alpha\in I$,

(ii) if $\tilde{Z}$ is another random variable satisfying (i) in the place of $\overline{Z}$, then $\mathbb{P}(\overline{Z}\le\tilde{Z})=1$.

The random variable $\overline{Z}$ is called the *essential supremum* of $(Z_\alpha)_{\alpha\in I}$ and is denoted by $\operatorname{ess\,sup}_{\alpha\in I} Z_\alpha$. In addition, if $\{Z_\alpha : \alpha \in I\}$ is upwards directed in the sense that for any $\alpha$, $\beta \in I$ there is $\gamma \in I$ such that $\max\{Z_\alpha, Z_\beta\} \le Z_\gamma$, then the countable set $J = \{\alpha_1, \alpha_2, \ldots\}$ can be chosen so that $Z_{\alpha_1} \le Z_{\alpha_2} \le \ldots$ and $\operatorname{ess\,sup}_{\alpha\in I} Z_\alpha = \lim_{n\to\infty} Z_{\alpha_n}$.

Now we see that (2.6) and (2.7) give the identity

$$B_n^N = \operatorname*{ess\,sup}_{\tau\in\mathcal{M}_n^N} \mathbb{E}(G_\tau|\mathcal{F}_n) \tag{2.8}$$

with probability 1. A nice feature of this alternative characterization of the sequence $(B_n^N)_{n=0}^N$ is that it extends naturally to the setting of infinite horizon (i.e., for $N = \infty$) and, as we shall prove now, provides the desired solution.

So, consider the optimal stopping problem (2.3) for $N = \infty$:

$$V_n = \sup_{\tau\ge n} \mathbb{E}G_\tau. \tag{2.9}$$

For $n = 0, 1, 2, \ldots$, introduce the random variable

$$B_n = \operatorname*{ess\,sup}_{\tau\ge n} \mathbb{E}(G_\tau|\mathcal{F}_n) \tag{2.10}$$

and the stopping time

$$\tau_n = \inf\{k \ge n : B_k = G_k\}, \tag{2.11}$$

with the usual convention $\inf\emptyset = \infty$. In the literature, the sequence $(B_n)_{n\ge0}$ is often referred to as the Snell envelope of $G$.

We will establish the following analogue of Theorem 2.1.

THEOREM 2.3. *Suppose that the sequence $(G_n)_{n\ge0}$ satisfies $\mathbb{E}\sup_{n\ge0}|G_n| < \infty$ and consider the optimal stopping problem (2.9). Then the following statements hold true.*

*(i) For any $n \ge 0$ we have the recurrence relation*

$$B_n = \max(G_n, \mathbb{E}(B_{n+1}|\mathcal{F}_n)).$$

*(ii) We have $\mathbb{P}(B_n \ge \mathbb{E}(G_\tau|\mathcal{F}_n)) = 1$ for all $\tau \in \mathcal{M}_n$ and, if the stopping time $\tau_n$ is finite almost surely, then $\mathbb{P}(B_n = \mathbb{E}(G_{\tau_n}|\mathcal{F}_n)) = 1$.*

*(iii) If $\mathbb{P}(\tau_n < \infty) = 1$, then $\tau_n$ is optimal in (2.9). Furthermore, if $\tau_*$ is another optimal stopping time for (2.9), then $\tau_n \le \tau_*$ almost surely.*

*(iv) The sequence $(B_k)_{k\ge n}$ is the smallest supermartingale which majorizes $(G_k)_{k\ge n}$. Moreover, the stopped process $(B_{\tau_n\wedge k})_{k\ge n}$ is a martingale.*

PROOF. We will only establish (i), the other parts can be shown with the argumentation similar to that used in the proof of Theorem 2.1. We need to show two inequalities to prove the identity. Take $\tau \in \mathcal{M}_n$ and let $\tau' = \tau \vee (n+1)$. Then $\tau' \in \mathcal{M}_{n+1}$ and since $\{\tau \ge n+1\} \in \mathcal{F}_n$, we may write

$$\begin{aligned}
\mathbb{E}(G_\tau|\mathcal{F}_n) &= \mathbb{E}(G_\tau 1_{\{\tau=n\}}|\mathcal{F}_n) + \mathbb{E}(G_\tau 1_{\{\tau\ge n+1\}}|\mathcal{F}_n)\\
&= 1_{\{\tau=n\}}G_n + 1_{\{\tau\ge n+1\}}\mathbb{E}(G_{\tau'}|\mathcal{F}_n)\\
&= 1_{\{\tau=n\}}G_n + 1_{\{\tau\ge n+1\}}\mathbb{E}\big[\mathbb{E}(G_{\tau'}|\mathcal{F}_{n+1})|\mathcal{F}_n\big]\\
&\le 1_{\{\tau=n\}}G_n + 1_{\{\tau\ge n+1\}}\mathbb{E}(B_{n+1}|\mathcal{F}_n)\\
&\le \max\{G_n, \mathbb{E}(B_{n+1}|\mathcal{F}_n)\}.
\end{aligned}$$

This proves the inequality "$\leq$". To show the reverse, observe that the family $\{\mathbb{E}(B_\tau|\mathcal{F}_{n+1}) \ : \ \tau \in \mathcal{M}_{n+1}\}$ is upwards directed. Indeed, if $\alpha$, $\beta \in \mathcal{M}_{n+1}$ and we set $\gamma = \alpha 1_A + \beta 1_{\Omega\setminus A}$, where $A = \{\mathbb{E}(G_\alpha|\mathcal{F}_{n+1}) \geq \mathbb{E}(G_\beta|\mathcal{F}_{n+1})\}$, then $\gamma$ is a stopping time belonging to $\mathcal{M}_{n+1}$ and

$$\begin{aligned}\mathbb{E}(G_\gamma|\mathcal{F}_{n+1}) &= \mathbb{E}(G_\alpha 1_A + G_\beta 1_{\Omega\setminus A}|\mathcal{F}_{n+1}) \\ &= 1_A\mathbb{E}(G_\alpha|\mathcal{F}_{n+1}) + 1_{\Omega\setminus A}\mathbb{E}(G_\beta|\mathcal{F}_{n+1}) \\ &= \max\big\{\mathbb{E}(G_\alpha|\mathcal{F}_{n+1}), \mathbb{E}(G_\beta|\mathcal{F}_{n+1})\big\}.\end{aligned}$$

Therefore, there is a sequence $\{\sigma_k : k \geq 1\}$ in $\mathcal{M}_{n+1}$ such that

$$\operatorname*{ess\,sup}_{\tau\in\mathcal{M}_{n+1}} \mathbb{E}(G_\tau|\mathcal{F}_{n+1}) = \lim_{k\to\infty} \mathbb{E}(G_{\sigma_k}|\mathcal{F}_{n+1})$$

and $\mathbb{E}(G_{\sigma_1}|\mathcal{F}_{n+1}) \leq \mathbb{E}(G_{\sigma_2}|\mathcal{F}_{n+1}) \leq \ldots$ with probability 1. Now we can write, by Lebesgue's monotone convergence theorem,

$$\begin{aligned}\mathbb{E}(B_{n+1}|\mathcal{F}_n) &= \mathbb{E}\left(\operatorname*{ess\,sup}_{\tau\in\mathcal{M}_{n+1}} \mathbb{E}(G_\tau|\mathcal{F}_{n+1})\middle|\mathcal{F}_n\right) \\ &= \mathbb{E}\left(\lim_{k\to\infty} \mathbb{E}(G_{\sigma_k}|\mathcal{F}_{n+1})\middle|\mathcal{F}_n\right) \\ &= \lim_{k\to\infty} \mathbb{E}\big(\mathbb{E}(G_{\sigma_k}|\mathcal{F}_{n+1})|\mathcal{F}_n\big) = \lim_{k\to\infty} \mathbb{E}(G_{\sigma_k}|\mathcal{F}_n) \leq B_n.\end{aligned}$$

Since $B_n \geq G_n$ (which can be trivially obtained by considering $\tau \equiv n$ in the definition of $B_n$), we get the desired identity. $\square$

In the remaining part of this subsection, let us inspect the connection between the contexts of finite and infinite horizons. One easily checks that the random variables $B_n^N$ and $\tau_n^N$ do not decrease as we increase $N$. Consequently, the limits

$$B_n^\infty := \lim_{N\to\infty} B_n^N \qquad \text{and} \qquad \tau_n^\infty := \lim_{N\to\infty} \tau_n^N$$

exist on a set of full measure. Furthermore, we also see that the sequence $(V_n^N)_{N=n}^\infty$ is nondecreasing, so the quantity $V_n^\infty = \lim_{N\to\infty} V_n^\infty$ is well-defined. Now it follows directly from (2.5), (2.8), (2.10) and (2.11) that

$$B_n^\infty \leq B_n \qquad \text{and} \qquad \tau_n^\infty \leq \tau_n \tag{2.12}$$

almost surely. Therefore, we also have

$$V_n^\infty \leq V_n. \tag{2.13}$$

THEOREM 2.4. *Suppose that* $\mathbb{E}\sup_{n\geq 0}|G_n| < \infty$ *and consider the optimal stopping problems* (2.3) *and* (2.9). *Then equalities hold in* (2.12) *and* (2.13).

PROOF. Letting $N \to \infty$ in the recurrence relation (2.4) yields

$$B_n^\infty = \max\{G_n, \mathbb{E}(B_{n+1}^\infty|\mathcal{F}_n)\}, \qquad n = 0,\, 1,\, 2,\, \ldots,$$

by Lebesgue's monotone convergence theorem. Consequently, $(B_n^\infty)_{n\geq 0}$ is an adapted supermartingale dominating $(G_n)_{n\geq 0}$. Thus $B_n^\infty \geq B_n$ for each $n$, by the fourth part of the preceding theorem. This shows the identity $B^\infty = B$ almost surely, and the remaining equalities follow immediately. $\square$

EXAMPLE 2.5. Strict inequalities may hold in (2.12) and (2.13) if the integrability condition on $\sup_{n\geq 0}|G_n|$ is not imposed. To see this, let $\varepsilon_0, \varepsilon_1, \varepsilon_2, \ldots$ be a sequence of independent Rademacher variables and set $G_n = \varepsilon_0 + \varepsilon_1 + \ldots + \varepsilon_n$. Then the process $(G_n)_{n\geq 0}$ is a martingale with respect to the natural filtration, so $V_n^N = 0$, $B_n^N = G_n$ and $\tau_n^N = n$ for all $0 \leq n \leq N < \infty$. Consequently, these identities are preserved in the limit: $V_n^\infty = 0$, $B_n^\infty = G_n$ and $\tau_n^\infty = n$ for all $n$. On the other hand, it is well-known that for any positive integer $a$, the stopping time $\sigma_n = \inf\{k \geq n : G_k = a\}$ is finite almost surely and hence $V_n \geq \mathbb{E}G_{\sigma_n} = a$. Since $a$ was arbitrary, we get $V_n = \infty$, $B_n = \infty$ and $\tau_n = \infty$ with probability 1.

**1.4. An example.** Let $\xi_0, \xi_1, \xi_2, \ldots$ be i.i.d. random variables following the $Exp(1)$ law, and let $c > 0$ be a fixed constant. We will solve the optimal stopping problem

$$V = \sup_{\tau\in\mathcal{M}} \mathbb{E}\Big[\max\{\xi_0, \xi_1, \xi_2, \ldots, \xi_\tau\} - c\tau\Big].$$

For the sake of clarity, we will split the reasoning into three separate steps. To put this problem into the general framework of the optimal stopping theory, we set

$$X_n = \max\{\xi_0, \xi_1, \xi_2, \ldots, \xi_n\} - cn.$$

Then $V = \sup_{\tau\in\mathcal{M}} X_\tau$ and we may proceed.

STEP 1. GUESSING THE OPTIMAL STOPPING RULE AND THE ASSOCIATED EXPECTATION. This is an informal step and it requires some thought and experimentation. It seems reasonable to conjecture that the optimal stopping rule should be of the following threshold type:

$$\tau_a = \inf\{n : \xi_n \geq a\}$$

for some unknown constant $a$. To find $a$, let us first compute the corresponding expectation

$$\mathbb{E}\Big[\max\{\xi_0, \xi_1, \xi_2, \ldots, \xi_{\tau_a}\} - c\tau_a\Big]. \tag{2.14}$$

Note that $\tau_a$ has the geometric distribution with parameter $\mathbb{P}(\xi_0 \geq a)$ and hence

$$\mathbb{E}\tau_a = \frac{\mathbb{P}(\xi_0 < a)}{\mathbb{P}(\xi_0 \geq a)} = \frac{1 - e^{-a}}{e^{-a}} = e^a - 1.$$

Furthermore, we have

$$\begin{aligned}\mathbb{E}\max\{\xi_0, \xi_1, \xi_2, \ldots, \xi_{\tau_a}\} &= \mathbb{E}\xi_{\tau_a} = \mathbb{E}(\xi_0|\xi_0 \geq a)\\ &= \frac{1}{\mathbb{P}(\xi_0 \geq a)}\int_{\{\xi_0\geq a\}} \xi_0 \mathrm{d}\mathbb{P}\\ &= e^a\left(ae^{-a} + 1\cdot e^{-a}\right) = a + 1\end{aligned}$$

and hence the expectation (2.14) equals $a + 1 - c(e^a - 1)$. We want to maximize this expectation (over all possible stopping times, and so, in particular, over $\tau_a$): we easily check that

$$\max_a \Big\{a + 1 - c(e^a - 1)\Big\} = \begin{cases} 1 & \text{if } c \geq 1 \text{ (maximum attained at } a_* = 0),\\ c - \ln c & \text{if } c < 1 \text{ (maximum attained at } a_* = -\ln c).\end{cases}$$

Let us denote the right-hand side by $\tilde{V}$. This is the candidate for the value of our optimal stopping problem.

STEP 2. GUESSING THE SNELL ENVELOPE. Actually, the computation from the previous step easily yields the corresponding candidate for the Snell envelope. From the general theory, we know that

$$B_n = \operatorname*{esssup}_{\tau \geq n} \mathbb{E}(X_\tau | \mathcal{F}_n).$$

Take $\tau = \tau_{a_*} \vee n$ (the additional maximum with $n$ is to enforce the estimate $\tau \geq n$): by the above computations, for this special stopping time, we have

$$\mathbb{E}(G_\tau|\mathcal{F}_n) = \begin{cases} \max\{\xi_0, \xi_1, \ldots, \xi_n\} - cn & \text{if } \max\{\xi_0, \xi_1, \ldots, \xi_n\} \geq a_*, \\ \tilde{V} - c(n+1) & \text{if } \max\{\xi_0, \xi_1, \ldots, \xi_n\} < a_*. \end{cases}$$

Let us denote the right-hand side by $\tilde{B}_n$: this is our candidate for the Snell envelope of $(G_n)_{n\geq 0}$. By the very definition, we have $\tilde{B}_n \leq B_n$.

STEP 3. VERIFICATION OF THE PROPERTIES OF $\tilde{B}$. Now we will check that $(\tilde{B}_n)_{n\geq 0}$ is a supermartingale majorizing $(G_n)_{n\geq 0}$. Then by the general theory we will obtain the reverse estimate $\tilde{B}_n \geq B_n$, which will show that $\tilde{B}$ coincides with the Snell envelope and the stopping time $\tau_{a_*}$ is optimal.

We start with the majorization. On the set $\{\max\{\xi_0, \xi_1, \ldots, \xi_n\} \geq a_*\}$ we have $\tilde{B}_n = G_n$. On the other hand, on $\{\max\{\xi_0, \xi_1, \ldots, \xi_n\} < a_*\}$ (which is nonempty iff $a_* > 0$, i.e., $c < 1$), the majorization is equivalent to

$$\tilde{V} - c(n+1) \geq \max\{\xi_0, \xi_1, \ldots, \xi_n\} - cn$$

or $-\ln c \geq \max\{\xi_0, \xi_1, \ldots, \xi_n\}$: but this is trivial, since $-\ln c = a_*$.

It remains to check the supermartingale property of $\tilde{B}_n$:

$$\mathbb{E}(\tilde{B}_{n+1}|\mathcal{F}_n) \leq \tilde{B}_n, \qquad n = 0,\, 1,\, 2,\, \ldots. \tag{2.15}$$

On the set $\{\max\{\xi_0, \xi_1, \ldots, \xi_n\} \geq a_*\} \in \mathcal{F}_n$ we automatically have the bound $\max\{\xi_0, \xi_1, \ldots, \xi_{n+1}\} \geq a_*$ and hence

$$\mathbb{E}(\tilde{B}_{n+1}|\mathcal{F}_n) = \mathbb{E}\Big( \max\{\xi_0, \xi_1, \ldots, \xi_{n+1}\} - c(n+1)|\mathcal{F}_n \Big).$$

Now, consider the random variable $\xi = \max\{\xi_0, \xi_1, \ldots, \xi_n\}$. But for any $a > a_*$,

$$\mathbb{E}\Big( \max\{a, \xi_{n+1}\} - c(n+1) \Big) \leq a - cn$$

(this is equivalent to $e^{-a} \leq c$ and holds true, since $e^{-a_*} \leq c$). Hence

$$\begin{aligned}\mathbb{E}\Big( \max\{\xi_0, \xi_1, \ldots, \xi_{n+1}\} - c(n+1)|\mathcal{F}_n \Big) &= \mathbb{E}\Big( \max\{a, \xi_{n+1}\} - c(n+1) \Big)\Big|_{a=\xi} \\ &\leq \xi - c_n = \tilde{B}_n.\end{aligned}$$

Next, we analyze (2.15) on the set $\{\max\{\xi_0, \xi_1, \ldots, \xi_n\} < a_*\}$ (which is nonempty iff $a_* > 0$, i.e., $c < 1$). On this set, we have the identity

$$\tilde{B}_{n+1} = \begin{cases} \xi_{n+1} - c(n+1) & \text{if } \xi_{n+1} \geq a_*, \\ \tilde{V} - c(n+2) & \text{if } \xi_{n+1} < a_*, \end{cases}$$

which is independent of $\mathcal{F}_n$. Consequently, the conditional expectation $\mathbb{E}(\tilde{B}_{n+1}|\mathcal{F}_n)$ is equal to the average of the right-hand side above, i.e., to

$$\begin{aligned}&\mathbb{E}\Big[(\xi_{n+1}-c(n+1))1_{\{\xi_{n+1}\geq a_*\}}+(\tilde{V}-c(n+2))1_{\{\xi_{n+1}<a_*\}}\Big]\\&=\mathbb{E}\xi_{n+1}1_{\{\xi_{n+1}\geq a_*\}}-c(n+1)+(\tilde{V}-c)\mathbb{P}(\xi_{n+1}<a_*)\\&=e^{-a_*}(a_*+1)-c(n+1)+(\tilde{V}-c)(1-e^{-a_*})=V-c(n+1)=\tilde{B}_n,\end{aligned}$$

where we have used the identity $a_*=-\ln c$.

This completes the proof of (2.15) and finishes the analysis of the optimal stopping problem.

## 2. Markovian approach

Throughout this section, we assume that $X=(X_0,X_1,X_2,\ldots)$ is a Markov family defined on $(\Omega,\mathcal{F},(\mathcal{F}_n)_{n\geq 0},(P_x)_{x\in E})$, taking values in some topological space $(E,\mathcal{B}(E))$. For the sake of simplicity, we will assume that $E=\mathbb{R}^d$ for some $d\geq 1$, though the reasoning remains essentially the same for other topological spaces. As usual, we assume that for each $x\in E$, we have $X_0\equiv x$ $\mathbb{P}_x$-almost surely. Let us also introduce the transition operator $T$ of $X$, which acts by the formula

$$Tf(x)=\mathbb{E}_xf(X_1)\qquad\text{for } x\in E,$$

on the class $I$ of all measurable functions $f:E\to\mathbb{R}$ such that $f(X_1)$ is $\mathbb{P}_x$-integrable for all $x\in E$.

Suppose that $N$ is a nonnegative integer and let $G:E\to\mathbb{R}$ be a measurable function satisfying

$$\mathbb{E}_x\left(\sup_{0\leq n\leq N}|G(X_n)|\right)<\infty\qquad\text{for all } x\in E. \tag{2.16}$$

Consider the associated finite-horizon optimal stopping problem

$$V^N(x)=\sup\mathbb{E}_xG(X_\tau), \tag{2.17}$$

where $x\in E$ and the supremum is taken over all $\tau\in\mathcal{M}^N$. Obviously, if we define $G_n=G(X_n)$ for $n=0$, 1, 2, …, then for each separate $x$ this problem is of the form considered in the preceding sections (with $\mathbb{P}$ and $\mathbb{E}$ replaced by $\mathbb{P}_x$ and $\mathbb{E}_x$). However, the joint study of the whole family of optimal stopping problems depending on the initial value $x$ enables the exploitation of the additional Markovian structure of the sequence $(X_n)_{n\geq 0}$.

For a given $x$, let us consider the random variables $B_n^N$ and the stopping times $\tau_n^N$, $n=0,1,2,\ldots,N$, defined by (2.4) and (2.5). We also introduce the sets

$$\begin{aligned}C_n&=\{x\in E\,:\,V^{N-n}(x)>G(x)\},\\D_n&=\{x\in E\,:\,V^{N-n}(x)=G(x)\},\end{aligned}$$

for $n=0,1,2,\ldots,N$; we will call these the continuation and stopping regions, respectively. Finally, define the stopping time

$$\tau_D=\inf\{0\leq n\leq N\,:\,X_n\in D_n\}.$$

Since $V^0=G$, by the very definition (2.17), we see that $X_N\in D_N$ and hence the stopping time $\tau_D$ is finite (it does not exceed $N$).

THEOREM 2.6. *Assume that the function $G$ satisfies the integrability condition* (2.16) *and consider the optimal stopping problem* (2.17).

*(i) For any $n = 0, 1, 2, \ldots, N$ we have $B_n^N = V^{N-n}(X_n)$.*

*(ii) The function $x \mapsto V^n(x)$ satisfies the Wald-Bellman equation*

$$V^n(x) = \max\{G(x), TV^{n-1}(x)\}, \qquad x \in E, \tag{2.18}$$

*for $n = 1, 2, \ldots, N$.*

*(iii) The stopping time $\tau_D$ is optimal in* (2.17). *If $\tau_*$ is another optimal stopping time, then $\tau_D \le \tau_*$ $\mathbb{P}_x$-almost surely for all $x \in E$.*

*(iv) For each $x \in E$, the sequence $(V^{N-n}(X_n))_{n=0}^N$ is the smallest $\mathbb{P}_x$-supermartingale majorizing $(G(X_n))_{n=0}^N$, and the stopped sequence $\big(V^{N-n\wedge\tau_D}(X_{n\wedge\tau_D})\big)_{n=0}^N$ is a $\mathbb{P}_x$-martingale.*

PROOF. We only need to establish (i) and (ii); the remaining parts follow at once from Theorem 2.1. To verify (i), recall that

$$B_n^N = \mathbb{E}_x\left[G(X_{\tau_n^N})|\mathcal{F}_n\right]$$

for all $n = 0, 1, 2, \ldots, N$. This shows the claim for $n = 0$, by the very definition of $V^N(x)$. On the other hand, for $n \ge 1$ we apply the Markov property to get

$$B_n^N = \mathbb{E}_y\left[G(X_{\tau_0^{N-n}})\right]\Bigg|_{y=X_n} = V^{N-n}(y)\big|_{y=X_n} = V^{N-n}(X_n).$$

(ii) We apply the definition of the sequence $(B_n^N)_{n=0}^N$ and part (i) to obtain that $\mathbb{P}_x$-almost surely,

$$\begin{aligned} V^N(x) = V^N(X_0) = B_0^N &= \max\{G(X_0), \mathbb{E}_x(B_1^N|\mathcal{F}_0)\} \\ &= \max\{G(x), \mathbb{E}_x(V^{N-1}(X_1)|\mathcal{F}_0)\} \\ &= \max\{G(x), TV^{N-1}(x)\}. \end{aligned}$$

□

Part (ii) above gives the following iterative method of solving (2.17). Define the operator $Q$ acting on $f \in I$ by the formula

$$Qf(x) = \max\{G(x), Tf(x)\}, \qquad x \in E.$$

COROLLARY 2.7. *We have $V^N(x) = Q^N G(x)$ for all $x \in E$ and all integers $N$.*

Let us illustrate the above considerations by analyzing the following simple example.

EXAMPLE 2.8. Let $(S_n)_{n\ge0}$ be a symmetric random walk over the space $E = \{-2, -1, 0, 1, 2\}$ stopped at $\{-2, 2\}$. Clearly, $(S_n)_{n\ge0}$ is a Markov family on $E$. Set $G(x) = x^2(x+2)$ and consider the optimal stopping problem

$$V^N(x) = \sup_{\tau \le N} \mathbb{E}_x G(S_\tau), \qquad x \in E.$$

To treat the problem successfully, we compute the sequence $V^0$, $V^1$, $V^2$, $V^3$, .... Directly from (2.18), we have

$$\begin{aligned} V^n(x) &= \max\{G(x), TV^{n-1}(x)\} \\ &= \begin{cases} \max\{G(x), V^{n-1}(x)\} & \text{if } x \in \{-2, 2\}, \\ \max\left\{G(x), \frac{1}{2}\big(V^{n-1}(x-1) + V^{n-1}(x+1)\big)\right\} & \text{if } x \in \{-1, 0, 1\}. \end{cases} \end{aligned}$$

For notational simplicity, let us identify a function $f : E \to \mathbb{R}$ with the sequence of its values $f(-2)$, $f(-1)$, $f(0)$, $f(1)$, $f(2)$. Using the above recurrence, we compute that

$$\begin{array}{rlllll} V^0 = G: & 0, & 1, & 0, & 3, & 16, \\ V^1: & 0, & 1, & 2, & 8, & 16, \\ V^2: & 0, & 1, & 4\frac{1}{2}, & 9, & 16, \\ V^3: & 0, & 2\frac{1}{4}, & 5 & 10\frac{1}{4}, & 16, \\ V^4: & 0, & 2\frac{1}{2}, & 6\frac{1}{4}, & 10\frac{1}{2}, & 16, \\ \ldots \end{array}$$

and so on. Suppose that we want to solve the problem

$$V^4(x) = \sup_{\tau \leq 4} \mathbb{E}_x G(S_\tau), \qquad x \in E.$$

The value function $V^4$ has been derived above; to describe the optimal stopping strategy, let us write down the continuation and stopping regions $C_i$ and $D_i$, $i = 0, 1, 2, 3, 4$. Directly from the above formulas for $V^i$, we see that

$$\begin{array}{ll} C_0 = \{-1, 0, 1\}, & D_0 = \{-2, 2\}, \\ C_1 = \{-1, 0, 1\}, & D_1 = \{-2, 2\}, \\ C_2 = \{0, 1\}, & D_2 = \{-2, -1, 2\}, \\ C_3 = \{0, 1\}, & D_3 = \{-2, -1, 2\}, \\ C_4 = \emptyset & D_4 = \{-2, -1, 0, 1, 2\}. \end{array}$$

The optimal strategy is to wait for the first step $n$ at which we visit the corresponding stopping set $D_n$; then we stop the process ultimately.

We turn our attention to the case of infinite horizon, i.e., we consider the optimal stopping problem (or rather a family of optimal stopping problems)

$$V(x) = \sup \mathbb{E}_x G(X_\tau), \qquad x \in E, \tag{2.19}$$

where the supremum is taken over the class $\mathcal{M}$ of all adapted stopping times. Recall that the class $I$ consists of all measurable functions $f : E \to \mathbb{R}$ such that $f(X_1)$ is $\mathbb{P}_x$-integrable for all $x \in E$. The following notion will be crucial in our further considerations.

DEFINITION 2.2. The function $f \in I$ is called *superharmonic* (or *excessive*) if we have

$$Tf(x) \leq f(x) \qquad \text{for all } x \in E.$$

We have the following simple observation.

LEMMA 2.9. *The function $f \in I$ is superharmonic if and only if $(f(X_n))_{n\geq 0}$ is a supermartingale under each $\mathbb{P}_x$, $x \in E$.*

PROOF. If $f$ is superharmonic, then by Markov property,

$$\mathbb{E}_x(f(X_{n+1})|\mathcal{F}_n) = \mathbb{E}_y f(X_1)\big|_{y=X_n} = Tf(X_n) \leq f(X_n),$$

for each $n$. To show the reverse implication, observe that if $(f(X_n))_{n\geq 0}$ is a supermartingale under each $\mathbb{P}_x$, then in particular

$$Tf(x) = \mathbb{E}_x(f(X_1)|\mathcal{F}_0) \leq f(x). \quad \square$$

To formulate the main theorem, we introduce the corresponding continuation set $C$ and stopping set $D$ by

$$C = \{x \in E : V(x) > G(x)\},$$
$$D = \{x \in E : V(x) = G(x)\}.$$

Moreover, we define the stopping time $\tau_D = \inf\{n : X_n \in D\}$. In contrast to the case of finite horizon, this stopping time need not be finite (which will force us to impose some additional assumptions: see the statement below).

THEOREM 2.10. *Consider the optimal stopping problem* (2.19) *and assume that*

$$\mathbb{E}_x \sup_{n \geq 0} |G(X_n)| < \infty, \qquad x \in E. \tag{2.20}$$

*Then the following holds.*

*(i) The function $V$ satisfies the Wald-Bellman equation*

$$V(x) = \max\{G(x), TV(x)\}. \tag{2.21}$$

*(ii) If $\tau_D$ is finite $\mathbb{P}_x$-almost surely for all $x \in E$, then $\tau_D$ is the optimal stopping time. If $\tau_*$ is another optimal stopping time, then $\tau_* \geq \tau_D$ $\mathbb{P}_x$-almost surely.*

*(iii) The value function $V$ is the smallest superharmonic function which majorizes the gain function $G$ on $E$.*

*(iv) The stopped sequence $(V(X_{\tau_D \wedge n}))_{n \geq 0}$ is a $\mathbb{P}_x$-martingale for every $x \in E$.*

PROOF. This follows immediately from the case of finite horizon and the limit Theorem 2.4. □

Let us make here an important comment on the uniqueness of the solutions to the Wald-Bellman equations (2.18) and (2.21). Clearly, in the case of finite horizon there is only one solution: indeed, the starting function $V^0$ coincides with $G$ and the formula (2.18) produces a unique sequence $V^1, V^2, \dots, V^N$. In the case of infinite horizon, the situation is less transparent. For instance, if $G$ is a constant function, say, $G \equiv c$, then any constant function $V \equiv c'$ for some $c' \geq c$ satisfies the Wald-Bellman equation. However, any solution to (2.21) is a superharmonic function majorizing $G$, so part (iii) of Theorem 2.10 immediately yields the following "minimality principle".

COROLLARY 2.11. *The value function $V$ is the minimal solution to* (2.21).

EXAMPLE 2.12. Let us provide solution to the infinite-horizon version of Example 2.8. Under the notation used there, we study the optimal stopping problem

$$V(x) = \sup_{\tau \in \mathcal{M}} \mathbb{E} G(S_\tau), \qquad x \in E.$$

The function $G$ is bounded, so the integrability assumption of Theorem 2.10 is satisfied. Thus, we know that $V$ is the least superharmonic function which majorizes $G$: here the superharmonicity means that

$$V(x) \geq \frac{1}{2}\big(V(x-1) + V(x+1)\big), \qquad \text{for } x \in \{-1, 0, 1\}.$$

In other words, we search for the smallest concave function on $\{-2, -1, 0, 1, 2\}$ majorizing the function $G$. One easily checks that the function $x \mapsto 4(x+2)$ is concave (since it is linear), majorizes $G$ and coincides with $G$ at the endpoints $\pm 2$. Thus it is the smallest majorant of $G$ and hence it must be equal to the value function $V$.

EXAMPLE 2.13. Let $\xi_1$, $\xi_2$, $\dots$ be a sequence of i.i.d. random variables with the distribution given by $\mathbb{P}(\xi_i = 1) = p$, $\mathbb{P}(\xi_i = -1) = q$, where $p+q = 1$ and $p < q$. For a given integer $x$, define $S_n = x + \xi_1 + \xi_2 + \ldots + \xi_n$, $n = 0, 1, 2, \ldots$. Then the sequences $(S_n)_{n\geq 0}$ (with varying $x$) form a Markov family. Consider the optimal stopping problem

$$V(x) = \sup_{\tau\in\mathcal{M}} \mathbb{E}_x S_\tau^+, \qquad x \in E.$$

One easily checks the integrability assumption (2.20) (with $G(x) = x^+$) is satisfied. This follows from the well-known fact that

$$\mathbb{P}\big(\sup_{n\geq 0}(\xi_1 + \xi_2 + \ldots + \xi_n) \geq k\big) = \left(\frac{p}{q}\right)^k, \qquad k = 0, 1, 2, \ldots. \tag{2.22}$$

Thus, we need to find the least superharmonic majorant of $G$: $V$ is the least function on $\mathbb{Z}$ satisfying

$$V(x) = \max\left\{x^+, pV(x+1) + qV(x-1)\right\}, \qquad x \in \mathbb{Z}.$$

To identify this object, let us try to inspect the properties of the continuation set $C$ and the stopping region $D$. A little thought suggests that these sets should be of the form $C = \{\ldots, b-2, b-1\}$, $D = \{b, b+1, \ldots\}$ for some positive integer $b$ (possibly infinite). While this is more or less clear by some intuitive argumentation, we should point out here that this can also be shown rigorously. Indeed, pick $x \in \mathbb{Z}_-$ and take the stopping time $\tau \equiv -x+1$. Then $V(x) \geq \mathbb{E}_x S_\tau^+ = p^{-x+1} > 0 = G(x)$, so in particular $C$ contains all nonpositive integers. Furthermore, if $x > 0$ lies in $C$, then so does $x-1$. To see this, note that for any $a \in \mathbb{Z}$ we have

$$(x+a)^+ - x^+ \leq (x-1+a)^+ - (x-1)^+$$

(which is equivalent to the trivial bound $(x+a)^+ \leq (x-1+a)^+ + 1$) and hence for any stopping time $\tau$, if we plug $a = \xi_1 + \xi_2 + \ldots + \xi_\tau$,

$$\mathbb{E}_x S_\tau^+ - G(x) \leq \mathbb{E}_{x-1} S_\tau^+ - G(x-1).$$

This yields

$$0 < V(x) - G(x) \leq V(x-1) - G(x-1) \tag{2.23}$$

and thus $x - 1 \in C$, as we have claimed. This shows that $C$ and $D$ are of the form postulated above and hence, by the general theory,

$$V(x) = \begin{cases} x & \text{if } x \geq b, \\ pV(x+1) + qV(x-1) & \text{if } x < b. \end{cases}$$

Let us first identify $V$ on $C$. Solving the linear recurrence, we check that

$$V(x) = \alpha + \beta\left(\frac{q}{p}\right)^x, \qquad x < b,$$

for some constants $\alpha$, $\beta \in \mathbb{R}$. It follows from (2.22) that $V(x) \to 0$ as $x \to -\infty$ (simply use the estimate $\mathbb{E}_x S_\tau^+ \leq \mathbb{E}_x \sup_{n\geq 0} S_n^+$): this implies $\alpha = 0$ and $\beta \geq 0$. This also shows that $b < \infty$. Indeed, otherwise $V(x)$ would explode exponentially as $x \to \infty$, but on the other hand, by (2.23), for $x > 0$ we would have

$$V(x) \leq G(x) + V(0) - G(0) = x + V(0) - G(0).$$

It remains to find $\beta$ and the boundary $b$. First, exploiting the Wald-Bellman equation, we see that $V(b-1) = pV(b) + qV(b-2)$. This implies $V(b) = \beta(q/p)^b$ and hence

$$V(x) = b\left(\frac{q}{p}\right)^{x-b} \qquad \text{for } x \le b.$$

Secondly, again by Wald-Bellman equation, we see that $V(b) \ge pV(b+1)+qV(b-1)$, which is equivalent to $b \ge p/(q-p)$. Finally, observe that if $x > b$, then

$$V(x) = x = px + qx > p(x+1) + q(x-1) = pV(x+1) + qV(x-1).$$

Therefore, if $b$ satisfies the inequality $b \ge p/(q-p)$, then the function

$$\mathcal{V}(x) = \begin{cases} x & \text{if } x \ge b, \\ b\,(q/p)^{x-b} & \text{if } x < b. \end{cases}$$

is excessive. Let us now check for which $b$ the inequality $\mathcal{V} \ge G$ holds. This majorization is clear on $\{b, b+1, b+2, \ldots\}$. Since the function $x \mapsto (q/p)^{x-b}$ is nonnegative, convex and coincides with $G$ at $x = b$, it suffices to check whether it is bigger than $G$ at $x = b-1$. The latter bound is equivalent to $b < q/(q-p) = p/(q-p)+1$. This actually *forces* us to take $b = \lceil p/(q-p)\rceil$: this is the only choice for the parameter such that the resulting function $\mathcal{V}$ is superharmonic and majorizes $G$. Summarizing, we have shown that

$$V(x) = \begin{cases} x & \text{if } x \ge \lceil p/(q-p)\rceil, \\ \lceil p/(q-p)\rceil\,(q/p)^{x-\lceil p/(q-p)\rceil} & \text{if } x < \lceil p/(q-p)\rceil. \end{cases}$$

Observe that by (2.22), the stopping time

$$\tau = \inf\Big\{n \,:\, S_n \ge \lceil p/(q-p)\rceil\Big\} \tag{2.24}$$

is infinite with positive probability. Therefore, there is no optimal stopping time $\tau^*$ which would be finite $\mathbb{P}_x$-almost surely for all $x$. Hence, the value function is attained asymptotically at the stopping times

$$\tau^{(M)} = \inf\Big\{n : S_n \notin \big[M, \lceil p/(q-p)\rceil\big]\Big\},$$

as $M \to -\infty$.

There is a different approach to the above problem, based on guessing. Namely, as we have already observed above, it seems plausible that the optimal stopping rule if of the form

$$\tau_b = \inf\big\{n : S_n \ge b\big\}$$

for some unknown constant $b \in \mathbb{Z}$. Let us directly compute the expectation $V_b(x) = \mathbb{E}_x S^+_{\tau_b}$. Clearly, if $b \le 0$, then $V_b(x) = x^+$; for $b > 0$, we exploit (2.22) to obtain

$$V_b(x) = \begin{cases} x & \text{if } x \ge b, \\ b\left(\frac{p}{q}\right)^{b-x} & \text{if } x < b. \end{cases}$$

In particular, $V_b(0) = 0$ for $b \le 0$ and $V_b(0) = b(p/q)^b$. Let us pick $b$ for which $V_b(0)$ is the largest: clearly, such a $b$ must be positive. For $b > 0$ we compute that

$$\frac{V_{b+1}(0)}{V_b(0)} = \frac{(b+1)p}{bq} > 1$$

if and only if $b < p/(q-p)$. Thus the maximal value of $V_b(0)$ is attained for $b = \lceil p/(q-p) \rceil$. This leads us to the *candidate*

$$\tilde{V}(x) = \begin{cases} x & \text{if } x \geq \lceil p/(q-p) \rceil, \\ \lceil p/(q-p) \rceil \left(\frac{p}{q}\right)^{\lceil p/(q-p) \rceil - x} & \text{if } x < \lceil p/(q-p) \rceil. \end{cases}$$

Note that $\tilde{V}(x) \leq V(x)$, since $\tilde{V}$ is built on examples of stopping times; to check the reverse bound, we verify that $\tilde{V}$ is an excessive majorant of $G$.

REMARK 2.14. Consider the following problem. Let $(S_n)_{n \geq 0}$ be a non-symmetric random walk as above, started at zero. We will identify the best constant $C$ such that

$$\mathbb{E} S_\tau \leq C \tag{2.25}$$

holds for all finite stopping times $\tau$.

The general Bellman function method asserts that in order to establish (2.25), it is enough to find a function $B : \mathbb{Z} \to \mathbb{R}$ satisfying the following properties:

1° $B(0) \leq C$;
2° $B(x) \geq G(x) = x^+$ for all $x \in \mathbb{Z}$;
3° We have $B(x) \geq qB(x-1) + pB(x+1)$ for all $x \in \mathbb{Z}$.

Indeed, having found such a $B$, we pick an arbitrary stopping time $\tau$ and note that $(B(S_{\tau \wedge n}))_{n \geq 0}$ is a supermartingale, by 3°. Therefore, by 2°, Doob's optional sampling and finally 1°, we obtain

$$\mathbb{E} S^+_{\tau \wedge n} \leq \mathbb{E} B(S_{\tau \wedge n}) \leq \mathbb{E} B(S_{\tau \wedge 0}) = B(0) \leq C. \tag{2.26}$$

Letting $n \to \infty$ and applying Fatou's lemma, we get (2.25).

How to find such a $B$? In the example above, we have constructed the *optimal* $B$ (i.e., the smallest one). It is not difficult to check that the alternative choice is

$$B(x) = \lceil p/(q-p) \rceil \left(\frac{p}{q}\right)^{\lceil p/(q-p) \rceil - x}$$

for all $x$. What about optimality of the constant? One searches for the stopping time such that in the chain (2.26) above, we obtain equalities. This leads to (2.24).

The final example illustrates the application of optimal stopping in the theory of sequential testing.

EXAMPLE 2.15. Let $p \in (0,1)$ be a fixed parameter. Suppose $X_1, X_2, \ldots$ are i.i.d. random variables drawn from an unknown distribution, which can be either $N(0,1)$ (with probability $p$) or $N(1,1)$ (with probability $1-p$). Consider the two hypotheses $H_k = \{X_j \sim N(k,1)\}$, $k = 0, 1$. Based on the observations of the consecutive variables, we must decide

· when to stop the observation;
· having stopped, which hypothesis to accept.

We assume that the cost of each observation is equal to $c$, furthermore, the cost incurred by accepting the wrong hypothesis is equal to 1.

*Step 1. Fitting into the scheme.* To solve this problem, let us first put it into the general framework developed above. Suppose that $\theta$ is a random variable with the distribution $\mathbb{P}(\theta = 0) = p$, $\mathbb{P}(\theta = 1) = 1 - p$ and let $\xi_1, \xi_2, \ldots$ be i.i.d. random variables with the distribution $N(0,1)$, independent of $\theta$. Let $X_j = \theta + \xi_j$. We want to compute

$$V = \inf_{\tau,d} \mathbb{E}\Big[c\tau + 1_{\{d=0,\theta=1\}} + 1_{\{d=1,\theta=0\}}\Big],$$

where $\tau$ is the stopping time of $X$ and $d$ is the decision rule, i.e., an $\mathcal{F}^X_\tau$-measurable random variable with values in $\{0,1\}$. This formulation does not quite fall into the scope of optimal stopping theory as the variable under the expectation are not $\mathcal{F}^X$-measurable. To handle this, consider the so-called *a posteriori process* $\pi_n = \mathbb{P}(\theta = 1|\mathcal{F}^X_n)$. By Bayes' theorem, denoting by $g_j$ the density of $N(j,1)$, we see that $\pi_n$ equals

$$\frac{(1-p)\prod_{j=1}^n g_1(X_j)}{p\prod_{j=1}^n g_0(X_j) + (1-p)\prod_{j=1}^n g_1(X_j)} = \frac{1}{\frac{p}{1-p} + \exp(X_1 + X_2 + \ldots + X_n - n/2)}.$$

Now, on the set $\{\tau = n\}$, the variable $d$ is $\mathcal{F}^X_n$-measurable, so

$$\mathbb{E}\big[1_{\{d=0,\theta=1\}} + 1_{\{d=1,\theta=0\}}|\mathcal{F}^X_n\big] = 1_{\{d=0\}}\pi_n + 1_{\{d=1\}}(1-\pi_n),$$

and hence the optimal rule $d$ is

$$d = \begin{cases} 0 & \text{if } \pi_n < 1-\pi_n \quad \text{(i.e., for } \pi_n < 1/2\text{)},\\ 1 & \text{if } \pi_n \geq 1-\pi_n \quad \text{(i.e., for } \pi_n \geq 1/2\text{)}.\end{cases}$$

Therefore,

$$V = \inf_\tau \mathbb{E}\Big[c\tau + \min\{\pi_\tau, 1-\pi_\tau\}\Big].$$

*Step 2. Solution.* One can check that the process $(\pi_n)_{n\geq 1}$ is Markovian, hence so is $((n,\pi_n))_{n\geq 0}$. We extend the optimal stopping problem to

$$\begin{aligned} V(n,s) &= \inf_\tau \mathbb{E}_{n,s}\Big[c\tau + \min\{\pi_\tau, 1-\pi_\tau\}\Big] \\ &= \inf_\tau \mathbb{E}_s\Big[c(n+\tau) + \min\{\pi_\tau, 1-\pi_\tau\}\Big] \\ &= cn + \inf_\tau \mathbb{E}\Big[c\tau + \min\{\pi_\tau, 1-\pi_\tau\}\Big] = cn + V(0,s).\end{aligned}$$

Clearly, the cost function $G(n,s) = cn + \min\{s, 1-s\}$ also enjoys this homogeneity. Now we will prove that $V$ is concave: for any $\lambda \in (0,1)$ and any $s_1$, $s_2 \in [0,1]$,

$$\lambda V(0,s_1) + (1-\lambda)V(0,s_2) \leq V\Big(0, \lambda s_1 + (1-\lambda)s_2\Big).$$

To see this, let us slightly complicate the context. Consider an auxiliary random variable $\eta$, such that $\mathbb{P}(\eta = s_1) = \lambda$, $\mathbb{P}(\eta = s_2) = 1-\lambda$, and suppose that $\theta$ is conditionally drawn as follows: if $\eta = s_j$, then $\mathbb{P}(\theta = 1) = s_j = 1 - \mathbb{P}(\theta = 0)$, $j = 1, 2$. Then the variables $\xi_1$, $\xi_2$, $\ldots$ are taken independently. Now, for an arbitrary stopping time $\tau$ of $X$ we have

$$\begin{aligned} &\mathbb{E}\Big[c\tau + \min\{\pi_\tau, 1-\pi_\tau\}\Big] \\ &= \sum_{j=1}^2 \mathbb{E}\Big[c\tau + \min\{\pi_\tau, 1-\pi_\tau\}|\eta = s_j\Big]\mathbb{P}(\eta = s_j) \geq \lambda V(0,s_1) + (1-\lambda)V(0,s_2).\end{aligned}$$

Taking the infimum over $\tau$ on the left-hand side, we obtain the concavity. Since $V(0,0) = G(0,0)$ and $V(1,0) = G(1,0)$ (by the very definition), we conclude that there are $\alpha < 1/2 < \beta$ such that $V = G$ on $[0, \alpha]$ and $[\beta, 1]$, and $V < G$ otherwise. The optimal stopping time $\tau_*$ is therefore given by

$$\tau_* = \inf\{n : \pi_n \leq \alpha \text{ or } \pi_n \geq \beta\}.$$

That is, if $\pi$ gets below $\alpha$, we accept $H_0$; otherwise, we accept $H_1$.

## 3. Problems

**1.** We flip a coin at most five times, at each point we may decide whether to stop or not (in particular, we are allowed to stop at the very beginning, without flipping the coin even once). Having stopped, we look at the outcomes we have obtained. We get 1 if there are no heads and get 2 if we obtained at least three heads. What is the strategy which yields the largest expected gain?

**2.** Let $G_1$, $G_2$, $\ldots$ be a sequence of independent random variables, each of which has the uniform distribution on $[0, 1]$. Solve the optimal stopping problems

$$V^N = \sup_{\tau \in \mathcal{M}^N} \mathbb{E} G_\tau \qquad \text{and} \qquad V_0 = \sup_{\tau \in \mathcal{M}} \mathbb{E} G_\tau,$$

where $N$ is an arbitrary integer.

**3.** Let $X_1$, $X_2$, $\ldots$, $X_N$, $Y_1$, $Y_2$, $\ldots$, $Y_N$ be independent random variables with uniform distribution on $[0, 1]$. Solve the optimal stopping problem

$$V^N = \sup \mathbb{E} X_\tau,$$

where the supremum is taken over all stopping times of $(\max\{X_n, Y_n\})_{n=1}^N$.

**4.** Solve the secretary problem.

**5.** Let $(S_n)_{n \geq 0}$ be a symmetric random walk over the integers and let $G(x) = \arctan x - x^-$, $x \in \mathbb{Z}$. Solve the optimal stopping problem

$$V(x) = \sup \mathbb{E}_x G(S_\tau), \qquad x \in \mathbb{Z},$$

where the supremum is taken over (i) all stopping times $\tau$, (ii) bounded stopping times $\tau$.

**6.** Suppose that $(G_n)_{n \geq 0}$ is an adapted sequence of random variables satisfying $\mathbb{E} \sup_{n \geq 0} |G_n| < \infty$ and let $(B_n)_{n \geq 0}$ be the associated Snell envelope. Prove the identity

$$\sup_{\tau \in \mathcal{M}} \mathbb{E} G_\tau = \sup_{\tau \in \mathcal{M}} \mathbb{E} B_\tau.$$

**7.** Let $G_0$, $G_1$, $G_2$, $\ldots$ be a family of arbitrarily dependent random variables taking values in $[0, 1]$. Prove the inequality

$$\mathbb{E} \sup_{n \geq 0} G_n \leq \sup_{\tau \in \mathcal{M}} \mathbb{E} G_\tau + \frac{1}{e}$$

and show that the constant $1/e$ cannot be decreased.

**8.** Solve the optimal stopping problem

$$\sup_\tau \mathbb{E}\big(Y_\tau^2 - \tau\big),$$

where $Y_0 = 10$ and $Y_{n+1} = Y_n + \xi_{n+1}$ for $n = 0, 1, 2, \ldots$. Here $(\xi_n)_{n \geq 1}$ is a sequence of independent random variables with the distribution

$$\mathbb{P}(\xi_n = 1) = 1/(4n) = 1 - \mathbb{P}(\xi_n = 0), \qquad n = 1, 2, \ldots.$$

**9.** We flip a coin infinitely many times. For $n \geq 1$, let $G_n = n2^n/(n+1)$ of there were no tails in the first $n$ flips, and $G_n = 0$ otherwise. Solve the optimal stopping problem

$$V = \sup_{\tau \in \mathcal{M}} \mathbb{E} G_\tau.$$

**10.** We toss a fair coin and for each $n \geq 0$, we denote by $X_n$ the length of the current sequence of consecutive tails after $n$ flips:

$$\overbrace{\ldots H \underbrace{TT \ldots T}_{X_n}}^{n \text{ flips}}.$$

Solve the optimal stopping problem

$$V = \sup_{\tau \in \mathcal{M}} \mathbb{E} \left( X_\tau - \frac{1}{16} \tau \right).$$

**11.** A pawn moves over the set $\{1, 2, \ldots, n\}$, according to the following rules. If at some time it is located at the point $k$, then at the next step it jumps, independently from its evolution in the past, to one of the points $k$, $k+1$, $\ldots$, $n$ (each choice has the same probability). Let $X_j$ be the location of the pawn at the time $j$. Assuming that $X_0 = 1$, describe the stopping time $\tau$ which maximizes the expectation $\mathbb{E} G(X_\tau)$, where $G(x) = x 1_{\{x<n\}}$.

**12.** Let $\varepsilon_1$, $\varepsilon_2$, $\ldots$ be the sequence of independent Rademacher variables and set $S_0 \equiv 0$ and $S_n = \varepsilon_1 + \varepsilon_2 + \ldots + \varepsilon_n$ for $n = 1, 2, \ldots$. Find the smallest constant $C$ such that for any stopping time $\tau$ adapted to the natural filtration of $S$ we have

$$\mathbb{E} S_\tau^4 \leq C \mathbb{E} \tau^2.$$

**13.** (R. Bellman [**1**]) At any time, a particle can be in one of two states, called $S$ and $T$. At the beginning, the particle is in the state $T$. We perform a sequence of operations of type $A$ and $L$. If the operation $A$ is used and the probability that the particle is in the state $T$ is equal to $x$, then this probability decreases to $ax$ ($0 < a < 1$ is a given constant). The operation $L$ consists of observing the particle and tells us definitely which state it is in. Given the strategy, let $\tau$ be the number of operations after which the particle is observed in $S$ (with certainty). Find the strategy giving the minimal expectation of $\tau$.

**14.** Let $X_1$, $X_2$, $\ldots$ be a sequence of i.i.d. random variables with the uniform distribution on $[0, 1]$. Solve the optimal stopping problem

$$V = \inf_\tau |X_1 + X_2 + \ldots + X_\tau - 1|.$$

CHAPTER 3

# Inequalities for martingale transforms and differentially subordinated processes

## 1. Description of the method

We turn our attention to another class of estimates arising in the probability theory and harmonic analysis. Though we will mainly focus on the martingale setting, we should emphasize that the results we will obtain have their analytic counterparts which can be expressed in terms of unconditional-type properties of the Haar system. Furthermore, the martingale inequalities which will be studied can be applied to obtain corresponding tight estimates for wide classes of Fourier multipliers. In other words, despite the probabilistic language, the contents of this chapter is meaningful from the point of view of harmonic analysis.

As we will see below, our approach to the estimates for martingale transforms/under the differential subordination is quite similar to that used in the previous chapter in the context of Markovian optimal stopping. We will use an analogous notation to make this similarity even more visible. We assume that $(\Omega, \mathcal{F}, \mathbb{P})$ is a probability space, filtered by $(\mathcal{F}_n)_{n\geq 0}$, a non-decreasing family of sub-$\sigma$-algebras of $\mathcal{F}$. Let $f = (f_n)_{n\geq 0}$, $g = (g_n)_{n\geq 0}$ be real-valued martingales, with the difference sequences $df = (df_n)_{n\geq 0}$, $dg = (dg_n)_{n\geq 0}$ given by

$$df_0 = f_0, \qquad df_n = f_n - f_{n-1}, \quad n = 1,\, 2,\, \ldots,$$

and similarly for $dg$. The martingale $f$ is *simple*, if for every $n$ the random variable $f_n$ takes only a finite number of values. We say that $g$ is a transform of $f$, if there is a predictable sequence $v = (v_n)_{n\geq 0}$ such that $dg_n = v_n df_n$ for each $n \geq 1$ (here by predictability of $v$ we mean that for any $n \geq 0$, the variable $v_n$ is $\mathcal{F}_{(n-1)\vee 0}$-measurable). This definition is slightly different from that used in the literature, where it is assumed that the equality $dg_n = v_n df_n$ holds also for $n = 0$ (in such a case, we will say that $g$ is a *full* transform of $f$ by $v$). We will say that $g$ is a (full) $\pm 1$-transform of $f$, if the transforming sequence $v$ takes values in the set $\{-1, 1\}$.

The central problem of this chapter can be formulated as follows. Let $G : \mathbb{R}^2 \to \mathbb{R}$ be a given function and suppose that we are interested in the inequality

$$\mathbb{E}G(f_n, g_n) \leq 0, \tag{3.1}$$

for all $n$ and all pairs $(f, g)$ of simple martingales such that $g$ is a full transform of $f$ by a predictable sequence bounded in absolute value by 1 (i.e., we assume that for any $n$, the random variable $v_n$ takes values in $[-1, 1]$). Note that we do not need to assume anything about the regularity of integrability of $G$: the fact that $f$ and $g$ are simple guarantees the existence of the expectation. Motivated by the results obtained in the preceding chapter, we may search, for each pair $(f, g)$ as above, for a supermartingale $(U_n)_{n\geq 0}$ majorizing $(G(f_n, g_n))_{n\geq 0}$ and satisfying

$U_0 \le 0$. Clearly, if such a supermartingale exists, then (3.1) holds true. The Bellman function approach rests on reducing the search to the supermartingales of the form $(U(f_n,g_n))_{n\ge 0}$ for some special function $U$ to be found; now the condition $U_0 \le 0$, the majorization $U_n \ge G(f_n,g_n)$ and the supermartingale property translate into appropriate pointwise properties of $U$. Specifically, consider the following conditions a function $U:\mathbb{R}^2 \to \mathbb{R}$ might satisfy.

1° (initial condition) We have $U(x,y) \le 0$ for all $|y| \le |x|$.

2° (majorization) We have $U \ge G$ on $\mathbb{R}^2$.

3° (concavity) The function $U$ is concave along any line of slope in $[-1,1]$.

The statement below describes the relation between the above set of conditions and the desired estimate (3.1).

Lemma 3.1. *If there is a function $U$ satisfying 1°, 2° and 3°, then* (3.1) *holds.*

Proof. Fix any pair $(f,g)$ of simple martingales such that $g$ is the full transform of $f$ by a certain predictable sequence $v$ bounded in absolute value by 1. The concavity condition implies that $(U(f_n,g_n))_{n\ge0}$ is a supermartingale: for any $n \ge 0$ we have

$$\begin{aligned}\mathbb{E}\big[U(f_{n+1},g_{n+1})|\mathcal{F}_n\big] &= \mathbb{E}\big[U(f_n+df_{n+1},g_n+dg_{n+1})|\mathcal{F}_n\big]\\ &= \mathbb{E}\big[U(f_n+df_{n+1},g_n+v_{n+1}df_{n+1})|\mathcal{F}_n\big]\\ &\le U(f_n,g_n),\end{aligned}$$

by 3° and the conditional version of Jensen's inequality. Therefore, applying 2° and then 1°, we obtain that for each $n$,

$$\mathbb{E}G(f_n,g_n) \le \mathbb{E}U(f_n,g_n) \le \mathbb{E}U(f_0,g_0) \le 0. \quad \square$$

There is a natural question about the existence of a special function $U$ and how to identify this object. To understand this issue, we will adopt the reasoning developed in the preceding chapter. Our starting observation is that the inequality (3.1) is closely related to the following optimal stopping problem

$$V^N = \sup \mathbb{E}G(f_N,g_N), \tag{3.2}$$

where the supremum is taken over all pairs $(f,g)$ of simple martingales such that $g$ is a full transform of $f$ by a predictable sequence bounded in absolute value by 1. Here the filtration is also allowed to vary as well as the probability space. At the first glance, the phrase "optimal stopping problem" seems a bit misleading here, since no stopping times enter the above problem. However, a little thought reveals that the stopping times actually do appear: one easily checks that if $g$ is a full transform of $f$ and $\tau \le N$ is a stopping time, then the stopped process $g^\tau$ is a full transform of $f^\tau$ (by the same sequence). Thus (3.2) can indeed be regarded as an optimal stopping problem, and the significant complication (in comparison to the preceding chapter) lies in the fact that we do not stop a given Markov process, but a much larger class of martingale pairs.

The problem (3.2) is of finite-horizon-type and admits a natural version for $N=\infty$:

$$V = V^\infty = \sup \mathbb{E}G(f_n,g_n), \tag{3.3}$$

where the supremum is taken over all $n$ and all pairs $(f,g)$ as above. Motivated by the theory of optimal stopping, our first step is to enlarge the class of problems so that the initial values of the martingales $f$, $g$ can be taken into account. Namely, for each $(x, y) \in \mathbb{R}$ and any $N$, set

$$V^N(x,y) = \sup \mathbb{E}G(f_N, g_N), \tag{3.4}$$

where the supremum is taken over all pairs $(f, g)$ of simple martingales starting from $(x, y)$ such that $g$ is a transform of $f$ by some predictable sequence bounded in absolute value by 1. An infinite-horizon version of (3.4) is introduced analogously:

$$V(x,y) = \sup \mathbb{E}G(f_n, g_n), \tag{3.5}$$

where the supremum is taken over all $n$ and all pairs $(f, g)$ as above.

THEOREM 3.2. *Consider the finite-horizon problem* (3.4). *The sequence* $(V^n)_{n\geq 0}$ *can be computed inductively from the relations* $V^0 = G$ *and, for* $n \geq 1$,

$$V^n(x,y) = \sup \Big\{ \alpha_1 V^{n-1}(x+t_1, y+at_1) + \alpha_2 V^{n-1}(x+t_2, y+at_2) \Big\}, \tag{3.6}$$

*where the supremum is taken over all numbers* $\alpha_1$, $\alpha_2 \geq 0$, $t_1$, $t_2 \in \mathbb{R}$ *and* $a \in [-1, 1]$ *such that* $\alpha_1 + \alpha_2 = 1$ *and* $\alpha_1 t_1 + \alpha_2 t_2 = 0$.

The equality (3.6) can be regarded as a version of Wald-Bellman equation. It has a very nice geometric meaning: to find $V^n(x, y)$, we take an arbitrary $a \in [-1, 1]$ and consider the restriction of $V^{n-1}$ to the line of slope $a$ passing through the point $(x, y)$, i.e., the function $t \mapsto V^{n-1}(x+t, y+at)$. If $\zeta_a$ is the smallest concave function which majorizes this restriction, then $V^n(x, y)$ is the supremum of $\zeta_a(0)$ over all $a$. Directly from this interpretation, we see that (3.6) can be rewritten in the form

$$V^n(x,y) = \sup \mathbb{E}V^{n-1}(x+\xi, y+a\xi), \tag{3.7}$$

where the supremum is taken over all random variables $a$ with values in $[-1, 1]$ and all simple centered random variables $\xi$ satisfying $\mathbb{E}(\xi|a) = 0$.

PROOF. The equality $V^0 = G$ follows from the very definition of the sequence $(V^n)_{n\geq 0}$. To show the recurrence (3.6), fix $\alpha_1$, $\alpha_2$, $t_1$, $t_2$ and $a$ as in its statement and take any martingales $(f^1, g^1)$ and $(f^2, g^2)$ as in the definition of $V^{n-1}(x + t_1, y + at_1)$ and $V^{n-1}(x + t_2, y + at_2)$, respectively. Suppose that these pairs are given on two probability spaces $(\Omega, \mathcal{F}^1, \mathbb{P}^1)$ and $(\Omega^2, \mathcal{F}^2, \mathbb{P}^2)$. Let us glue these pairs into one pair $(f, g)$. To this end, let $\Omega = \Omega^1 \cup \Omega^2$, $\mathcal{F} = \sigma(\mathcal{F}^1 \cup \mathcal{F}^2)$ and define the probability measure $\mathbb{P}$ on $\mathcal{F}$ by requiring that $\mathbb{P}(A_1 \cup A_2) = \alpha_1 \mathbb{P}^1(A_1) + \alpha_2 \mathbb{P}^2(A_2)$ for any $A_1 \in \mathcal{F}^1$ and $A_2 \in \mathcal{F}^2$. The pair $(f, g)$ is given by $(f_0, g_0) = (x, y)$ and

$$(f_k(\omega), g_k(\omega)) = \begin{cases} (f^1_{k-1}(\omega), g^1_{k-1}(\omega)) & \text{if } \omega \in \Omega^1, \\ (f^2_{k-1}(\omega), g^2_{k-1}(\omega)) & \text{if } \omega \in \Omega^2. \end{cases}$$

Notice that the sequence $(f, g)$ is a martingale with respect to its natural filtration $(\mathcal{F}_n)_{n\geq 0}$: this follows at once from the fact that $(f^1, g^1)$, $(f^2, g^2)$ are martingales and the identity

$$\mathbb{E}((f_1, g_1)|\mathcal{F}_0) = \mathbb{E}(f_1, g_1) = \alpha_1(x + t_1, y + at_1) + \alpha_2(x + t_2, y + at_2) = (x, y).$$

Consequently, by the very definition of the functions $V^n$ and $V^{n-1}$,

$$V^n(x,y) \geq \mathbb{E}G(f_n, g_n) = \alpha_1 \mathbb{E}^1 G(f^1_{n-1}, g^1_{n-1}) + \alpha_2 \mathbb{E}^2 G(f^2_{n-1}, g^2_{n-1})$$

(where $\mathbb{E}^i$ is the expectation with respect to $\mathbb{P}^i$), so taking the supremum over all $(f^1, g^1)$, $(f^2, g^2)$ as above, we get

$$V^n(x,y) \geq \alpha_1 V^{n-1}(x+t_1, y+at_1) + \alpha_2 V^{n-1}(x+t_2, y+at_2).$$

Taking the supremum over all $\alpha_i$, $t_i$ and $a$, we obtain the inequality "$\geq$" in (3.6) and hence also in (3.7). To show the reverse estimate, take an arbitrary pair $(f, g)$ as in the definition of $V^n(x, y)$ and note that

$$V^{n-1}(f_1, g_1) \geq \mathbb{E}\big[G(f_n, g_n)|(f_1, g_1)\big],$$

by the very definition of $V^{n-1}$ applied conditionally on the $\sigma$-algebra generated by $(f_1, g_1)$. Therefore

$$\mathbb{E}G(f_n, g_n) = \mathbb{E}\left[\mathbb{E}(G(f_n, g_n)|(f_1, g_1)\right] \leq \mathbb{E}V^{n-1}(x + df_1, y + v_0 df_1)$$

does not exceed the right-hand side of (3.7). Taking the supremum over all $(f, g)$ we get the claim. □

The passage to the case of infinite horizon requires a simple limiting argument. Indeed, it follows directly from (3.2) and (3.3) that the functional sequence $(V^n)_{n\geq 0}$ is pointwise increasing (any martingale $f_0$, $f_1$, $f_2$, $\ldots$, $f_{n-1}$ of length $n$ can be treated as a martingale $f_0$, $f_1$, $f_2$, $\ldots$, $f_{n-1}$, $f_{n-1}$ of length $n$) and $V(x, y) = V^\infty(x, y) = \lim_{n\to\infty} V^n(x, y)$. Therefore, we obtain the following

Corollary 3.3. *Consider the problem* (3.5). *Then $V$ is the smallest function which satisfies the conditions 2° and 3°. Furthermore, if the inequality* (3.1) *is valid, then $V$ satisfies 1° as well.*

Proof. We have $V^n \geq V^0 = G$, so indeed $V$ majorizes $G$. Letting $n \to \infty$ in (3.6), we see that

$$V(x,y) = \sup\left\{\alpha_1 V(x+t_1, y+at_1) + \alpha_2 V(x+t_2, y+at_2\right\},$$

where the supremum is taken over all $t_i$, $\alpha_i$ and $a$ as above. This implies that $V$ has the required concavity property. To see that $V$ is the smallest, fix an arbitrary function $\mathcal{V}$ satisfying 2° and 3°, a point $(x, y)$ and a pair $(f, g)$ as in the definition of $V(x, y)$. As we have seen in the proof of Lemma 3.1, then the sequence $(\mathcal{V}(f_n, g_n))_{n\geq 0}$ is a supermartingale majorizing $(G(f_n, g_n))_{n\geq 0}$, so

$$\mathbb{E}G(f_n, g_n) \leq \mathbb{E}\mathcal{V}(f_n, g_n) \leq \mathbb{E}\mathcal{V}(f_0, g_0) = \mathcal{V}(x, y).$$

Taking the supremum over all $(f, g)$ yields the desired bound $V(x, y) \leq \mathcal{V}(x, y)$. It remains to show that if (3.1) holds true, then 1° holds. This follows immediately from the fact that if $|y| \leq |x|$, the martingale pair $(f, g)$ starts from $(x, y)$ and $g$ is a transform of $f$ by a predictable sequence $v$ bounded in absolute value by 1, then actually $g$ is a full transform of $f$ (with $v_0 \in [-1, 1]$ determined by the condition $y = v_0 x$). □

The above discussion concerns the case when the transforming sequence takes values in $[-1, 1]$. However, the approach can be easily modified to apply to other possibilities, e.g. to the case of $\pm 1$-transforms (when $v_n \in \{-1, 1\}$ for all $n$), or to the case when $v_n \in [0, 1]$ for all $n$, etc. Let us be more precise. Suppose we are interested in proving (3.1) for all $(f, g)$ such that $g$ is a full transform of $f$ by a predictable sequence $v$ taking values in some fixed set $A \subset \mathbb{R}$. A straightforward modification of the above arguments shows that the validity of this estimate is equivalent to the existence of a function $U$ satisfying

1° (initial condition) We have $U(x,y) \leq 0$ for all $(x,y)$ such that $y = vx$ for some $v \in A$.

2° (majorization) We have $U \geq G$ on $\mathbb{R}^2$.

3° (concavity) The function $U$ is concave along any line of slope in $A$.

An important observation is that the passage from the set $[-1,1]$ to $\{-1,1\}$ simplifies the analysis and in many cases does not reduce the validity of the results. More precisely, we will frequently perform the following procedure. Suppose that we want to establish (3.1) for all $f$, $g$ such that $g$ is a full transform of $f$ by a predictable sequence with values in $[-1,1]$. The first step is to consider the more restrictive (and a little simpler) case of $\pm 1$-transforms first and identify the corresponding special function $U$. The second step is to verify that this function $U$ actually satisfies all the properties needed for the more general case of $[-1,1]$-valued transforming sequences. This phenomenon occurs in most interesting estimates and is what one might expect: when studying the $[-1,1]$-case, it seems reasonable that the extremal martingales (for which the equality or almost equality occurs) should be constructed with the use of extremal transforming sequences, with values in $\{-1,1\}$. Analogous reasoning can be applied when reducing the case of $[0,1]$-valued to $\{0,1\}$-valued transforming sequences, and so on.

Actually, the special functions $U$ obtained via the analysis of $\pm 1$-transforms often lead to much wider class of inequalities for martingales satisfying the so-called differential subordination, a condition which is of significant importance from the point of view of applications. Here is the precise definition.

DEFINITION 3.1. A martingale $g$ is differentially subordinate to $f$ if for any $n \geq 0$ we have $|dg_n| \leq |df_n|$ almost surely.

Of course, if $g$ is a transform of $f$ by a predictable sequence with values in $[-1,1]$, then $g$ is differentially subordinate to $f$. However, the differential subordination allows a much wider class of martingale pairs $(f,g)$. Nevertheless, a similar approach to that used above can be used to the study of estimates in this new setting. Suppose that $G : \mathbb{R}^2 \to \mathbb{R}$ is a given function and assume we are interested in showing the inequality (3.1) for any pair $(f,g)$ of simple martingales such that $g$ is differentially subordinate to $f$. We should stress here that in a typical situation, the assumption on the simplicity of the sequences can be skipped, mostly often by imposing some boundedness-type conditions on $f$; for the sake of clarity, we will work with simple martingales only. Suppose that $U : \mathbb{R}^2 \to \mathbb{R}$ satisfies the following properties:

1° (initial condition) We have $U(x,y) \leq 0$ for all $(x,y)$ such that $|y| \leq |x|$.

2° (majorization) We have $U \geq G$ on $\mathbb{R}^2$.

3° (concavity) For any $(x,y) \in \mathbb{R}^2$ there are two numbers $A = A(x,y)$ and $B = B(x,y)$ such that if $h$, $k \in \mathbb{R}$ satisfy $|k| \leq |h|$, then

$$U(x+h, y+k) \leq U(x,y) + A(x,y)h + B(x,y)k. \tag{3.8}$$

As we have mentioned above, in many cases the special function obtained from the analysis of the corresponding estimate for $\pm 1$-transforms satisfies these conditions; in 3°, typically one takes the partial derivatives $A(x,y) = U_x(x,y)$ and $B(x,y) = U_y(x,y)$, or their one-sided versions.

LEMMA 3.4. *If there is a function $U$ satisfying 1°, 2° and 3°, then* (3.1) *is valid for any pair* $(f,g)$ *such that $g$ is differentially subordinate to $f$.*

PROOF. As in the case of transforms, the main ingredient of the proof is the supermartingale property of the sequence $(U(f_n,g_n))_{n\geq 0}$. Fix $n\geq 0$ and apply 3° to $x=f_n$, $y=g_n$, $h=df_n$ and $k=dg_n$ (the condition $|k|\leq|h|$ follows from the differential subordination) to obtain

$$U(f_{n+1},g_{n+1})\leq U(f_n,g_n)+A(f_n,g_n)dg_{n+1}+B(f_n,g_n)dg_{n+1}.$$

Taking the conditional expectation with respect to $\mathcal{F}_n$ yields the desired supermartingale property. The remainder of the proof is as previously:

$$\mathbb{E}G(f_n,g_n)\leq\mathbb{E}U(f_n,g_n)\leq\mathbb{E}U(f_0,g_0)\leq 0. \quad \square$$

The remaining part of this section is a list of useful comments and observations which will be used frequently in our later considerations.

(a) In general, the special function $U$ satisfying 1°, 2° and 3°, if it exists, is not unique. In some situations the choice of the appropriate function does simplify the calculations involved.

(b) In many cases, the special function inherits certain structural properties from $G$. For example, if $G$ is symmetric with respect to $x$-variable (i.e., we have $G(x,y)=G(-x,y)$ for all $x$, $y$), then we may search for $U$ in the class of functions enjoying this property. That is, if there is $U$ satisfying 1°, 2° and 3°, then there exists a function $\mathcal{U}$ which enjoys these conditions as well as the additional property $\mathcal{U}(x,y)=\mathcal{U}(-x,y)$ for all $x$, $y$. To see this, one easily verifies that

$$\mathcal{U}(x,y)=\min\{U(x,y),U(-x,y)\},$$

or

$$\mathcal{U}(x,y)=\frac{U(x,y)+U(-x,y)}{2},$$

is such a function. Alternatively, one can check that the function $V$ given by (3.5) meets this requirement (directly from the definition). Similarly, if $G$ is homogeneous of order $p$ and there exists a function $U$ satisfying 1°, 2° and 3°, then there is also a function $\mathcal{U}$ enjoying these properties which is homogeneous of order $p$. Indeed, it suffices to take

$$\mathcal{U}(x,y)=\inf_{\lambda>0}\left\{\lambda^pU(x/\lambda,y/\lambda)\right\},$$

or check that the solution to (3.5) has the required property.

(c) In the above discussion we have considered the real-valued martingales $f$ and $g$ only. This can be easily modified to the case when the sequences take values in some other domains. For instance, suppose we are interested in showing (3.1) under the assumption that $f$ takes values in $[0,1]$ and $g$ is its transform by a predictable sequence with values in $[-1,1]$ (in particular, $g$ *may* take negative values). Then only some minor straightforward modifications of the approach are required. Namely, one needs to construct a function on $[0,1]\times\mathbb{R}$ which satisfies

1° (initial condition) We have $U(x,y)\leq 0$ for all $|y|\leq x\leq 1$.

2° (majorization) We have $U\geq G$ on $[0,1]\times\mathbb{R}$.

3° (concavity) The function $U$ is concave along any line segment of slope in $[-1,1]$, contained in $[0,1]\times\mathbb{R}$.

Analogously, one can extend the method so that it works for Hilbert or Banach-space valued sequences. We will not pursue the discussion in this direction. See the bibliographical details at the end of the chapter.

(d) It should also be mentioned here that in the above martingale context, there are very natural analogues of the continuation and the stopping regions $C$ and $D$ which were so meaningful in the previous chapter. These sets can be defined with the same formula as in the theory of optimal stopping:

$$C = \{(x,y) \in R^2 : V(x,y) > G(x,y)\}, \qquad D = \{(x,y) \in \mathbb{R}^2 : V(x,y) = G(x,y)\}.$$

If one looks at the formula (3.5), these sets have very natural interpretation. Namely, if $(x,y)$ belongs to the set $D$, then the optimal choice for the pair $(f,g)$ (for which the supremum defining $V(x,y)$ is attained) is just the constant pair $(f,g) \equiv (x,y)$. In other words, when starting from the set $D$, it is optimal to "stop" the martingale pair instantly. In contrast, when $(x,y) \in C$, the optimal pair $(f,g)$ for $V(x,y)$ must make some nontrivial moves, that is, it must "continue" its evolution.

## 2. Examples

In this section we will illustrate the above method on several important examples. We assume throughout that all the processes we work with are simple.

EXAMPLE 3.5. Suppose that $g$ is differentially subordinate to $f$ and we are interested in the best constant $C$ in the inequality

$$\lambda \mathbb{P}(|g_n| \geq \lambda) \leq C\mathbb{E}|f_n|^2, \qquad n = 0,\, 1,\, 2,\, \ldots, \tag{3.9}$$

where $\lambda > 0$ is a fixed parameter. Note that we may assume that $\lambda = 1$, replacing $f$, $g$ with $f/\lambda$ and $g/\lambda$, respectively (this replacement does not affect the differential subordination). There are two objects to be determined: a priori unknown optimal value of the constant $C$ and an appropriate special Bellman function. As we have explained above, we first restrict ourselves to the case of $\pm 1$-transforms and hope that the function $U$ we will obtain will also work in the general setting. We rewrite the estimate in the form

$$\mathbb{E}G(f_n, g_n) \leq 0,$$

with $G(x,y) = 1_{\{|y|\geq 1\}} - C|x|^2$. We will identify the sequence $V^0$, $V^1$, $V^2$, .... By the very definition, $V^0 = G$. Note that the function $\mathcal{V}(x,y) = 1 - C|x|^2$ is concave; this, by a straightforward induction, implies $V^n \leq \mathcal{V}$ on $\mathbb{R}^2$. Since $G = \mathcal{V}$ outside the strip $\mathbb{R} \times (-1,1)$, we see that

$$V^n(x,y) = \mathcal{V}(x,y) = 1 - C|x|^2 \qquad \text{if } |y| \geq 1.$$

To find $V^n$ on the remaining part of the domain, fix $(x,y) \in \mathbb{R} \times (-1,1)$ and write down the "Wald-Bellman" equation for $V^1$:

$$V^1(x,y) = \sup\left\{\alpha_1 G(x+t_1, y+at_1) + \alpha_2 G(x+t_2, y+at_2)\right\}.$$

Here the supremum is taken over all $\alpha_i \geq 0$, $t_i \in \mathbb{R}$ and $a = \pm 1$ such that $\alpha_1 + \alpha_2 = 1$ and $\alpha_1 t_1 + \alpha_2 t_2 = 0$. So, we need to find the least concave majorant

$\zeta_1$ of $t \mapsto G(x+t, y+t)$, the least majorant $\zeta_{-1}$ of $t \mapsto G(x+t, y-t)$ and set $V^1(x,y) = \max\{\zeta_1(0), \zeta_{-1}(0)\}$. The function $\xi(t) = G(x+t, y+t)$ has the formula

$$\xi(t) = \begin{cases} -C(x+t)^2 & \text{if } t \in (-1-y, 1-y), \\ 1 - C(x+t)^2 & \text{otherwise} \end{cases}$$

and hence

$$\zeta_1(0) \geq \frac{1+y}{2}\xi(1-y) + \frac{1-y}{2}\xi(-1-y) = 1 - C + C(y^2 - x^2). \tag{3.10}$$

This inequality yields a useful lower bound for $C$. Namely, if we take $x = y \in (0,1)$, then, by 1°,

$$0 \geq V(x,y) \geq V^1(x,y) \geq 1 - C + C(y^2 - x^2) = 1 - C,$$

or $C \geq 1$. under this additional assumption, it is not difficult to check that

$$\zeta_1(t) = \begin{cases} 1 - C + C((y+t)^2 - (x+t)^2) & \text{if } t \in (-1-y, 1-y), \\ 1 - C(x+t)^2 & \text{otherwise,} \end{cases}$$

and

$$\zeta_{-1}(t) = \begin{cases} 1 - C + C((y-t)^2 - (x+t)^2) & \text{if } t \in (-1+y, 1+y), \\ 1 - C(x+t)^2 & \text{otherwise,} \end{cases}$$

so $V^1(x,y) = 1 - C + C(y^2 - x^2)$ for $(x,y) \in \mathbb{R} \times (-1,1)$. Now, one easily checks that $V^1$ is concave along the lines of slope $\pm 1$. This implies that the sequence $(V^n)_{n \geq 0}$ stabilizes: we have $V^1 = V^2 = \ldots = V$. Since $V^1(x,y) \leq 0$ for $y = \pm x$, we have proved the validity of (3.9) for any $C \geq 1$ (under the assumption that $g$ is a $\pm 1$-transform of $g$). Therefore, $C = 1$ is the best constant there.

Now we can pass to the more general classes of martingales. Setting $C = 1$, one can check that the function

$$V(x,y) = \begin{cases} y^2 - x^2 & \text{if } |y| < 1, \\ 1 - x^2 & \text{if } |y| \geq 1 \end{cases}$$

is concave along lines of slope belonging to $[-1,1]$ and satisfies $V(x,y) \leq 0$ for $|y| \leq |x|$. Therefore, we obtain that the inequality (3.9) holds for transforming sequences with values in $[-1,1]$. Actually, this inequality holds even in the less restrictive setting of differential subordination. Indeed, the appropriate versions of the initial and majorization conditions are satisfied, so it remains to verify the concavity inequality (3.8). We set $A(x,y) = -2x$, $B(x,y) = 2y1_{\{|y|<1\}}$ (note that these are essentially the partial derivatives of $V$) and consider two cases. If $|y| \geq 1$, we use the fact that the function $\mathcal{V}(x,y) = 1 - x^2$ is concave and majorizes $V$. Therefore, for *any* $h$, $k \in \mathbb{R}$,

$$\begin{aligned} V(x+h, y+k) &\leq \mathcal{V}(x+h, y+k) \\ &\leq \mathcal{V}(x,y) + \mathcal{V}_x(x,y)h + \mathcal{V}_y(x,y)k \\ &= V(x,y) + A(x,y)h + B(x,y)k. \end{aligned}$$

On the other hand, if $|y| < 1$ and $|k| \leq |h|$, then

$$\begin{aligned} V(x,y) + A(x,y)h + B(x,y)k &= (y+k)^2 - (x+h)^2 + h^2 - k^2 \\ &\geq (y+k)^2 - (x+h)^2 \\ &\geq V(x+h, y+k). \end{aligned}$$

This proves the validity of (3.9) under the assumption of the differential subordination of $g$ to $f$.

We conclude this lengthy analysis of the weak-type inequality by the observation that the function $V$ we used above is the smallest possible, but not the "simplest possible". There is a different, nicer choice for the function which yields the validity of (3.9): it is easy to see that $U(x,y) = y^2 - x^2$ has all the required properties. See the Remark (a) above.

However, in general the computation of the whole sequence $V^0$, $V^1$, $V^2$, ... is a formidable task and the calculations are hard to push through. Therefore, a typical approach rests on the direct search for the function $V$. Let us move to the next example.

EXAMPLE 3.6. We will identify the best constant $C_1$ in the estimate

$$\lambda \mathbb{P}(|g_n| \geq \lambda) \leq C_1 \mathbb{E}|f_n|, \qquad n = 0,\, 1,\, 2,\, \ldots,$$

where $\lambda > 0$ and $g$ is assumed to be differentially subordinate to $f$. As previously, we start the analysis in the more restrictive setting of $\pm 1$-transforms and assume that $\lambda = 1$. Then the problem can be rewritten in the form (3.1) with $G(x,y) = 1_{\{|y|\geq 1\}} - C_1|x|$. To gain some intuition about the special function to be found, let us write down the definition of $V$:

$$V(x,y) = \sup \big\{ \mathbb{P}(|g_n| \geq 1) - C_1 \mathbb{E}|f_n| \big\}, \tag{3.11}$$

the supremum taken over the usual parameters. We divide the analysis into a few intermediate steps.

*Step 1. The case* $|y| \geq 1$. We know that $V$ is the smallest majorant of $G$ which is concave along the lines of slope $\pm 1$. On the other hand, observe that the function $\mathcal{V}(x,y) = 1 - C_1|x|$ is concave and majorizes $G$: this implies $V \leq \mathcal{V}$ on the whole $\mathbb{R}^2$. Since $\mathcal{V}$ and $G$ coincide on $\{(x,y) : |y| \geq 1\}$, we must have

$$V(x,y) = G(x,y) = 1 - C_1|x| \qquad \text{provided } |y| \geq 1.$$

*Step 2. The case* $|x| + |y| \geq 1$. Actually, the above equality holds for all $x$, $y$ satisfying $|x| + |y| \geq 1$. This can be easily seen geometrically, by looking at the graph of the function $G$; the formal proof goes as follows. Suppose that $x$, $y \geq 0$ (for the remaining cases the reasoning is analogous) and $y < 1$. We have

$$V(x,y) \geq \frac{1-y}{2} V\left(x+y+1,-1\right) + \frac{1+y}{2} V\left(x+y-1,1\right) = 1 - C_1 x = \mathcal{V}(x,y).$$

Since $\mathcal{V} \geq V$ on the whole $\mathbb{R}^2$, we must actually have equality above.

*Step 3. The case* $|x| + |y| < 1$. Here we make a guess. Take a line of slope 1 passing through $(x,y)$: this line intersects the set $\{(x,y) : |x| + |y| = 1\}$ at two points

$$P_1 = \left( \frac{1+x-y}{2}, \frac{1-x+y}{2} \right), \qquad P_2 = \left( \frac{-1+x-y}{2}, \frac{-1-x+y}{2} \right).$$

We have, by the concavity of $V$,

$$V(x,y) \geq \frac{1+x+y}{2} V(P_1) + \frac{1-x-y}{2} V(P_2) = 1 - C_1(1 + x^2 - y^2)/2.$$

We assume that we actually have equality here. Since $V(0,0)$ must be nonpositive, this implies $C_1 \geq 2$ (note that this is a formal proof that the weak-type constant cannot be smaller than 2). This leads us to the candidate

$$U(x,y) = \begin{cases} 1 - C_1(1+x^2-y^2) & \text{if } |x|+|y|<1, \\ 1 - C_1|x| & \text{if } |x|+|y| \geq 1. \end{cases}$$

It is easy to check that this function has all the required properties and the inequality $\mathbb{P}(|g_n| \geq 1) \leq C_1 \mathbb{E}|f_n|$ is established (in the setting of $\pm 1$-transforms). Since $C_1 \geq 2$ was arbitrary, we see that the weak-type constant is equal to 2.

The passage to more general contexts of $[-1,1]$-valued transforming sequences and the setting of differential subordination goes along the same lines as previously. Clearly, it is enough to study the less restrictive context of differential subordination. Fix $C_1 = 2$. Then the function

$$U(x,y) = \begin{cases} y^2 - x^2 & \text{if } |x|+|y|<1, \\ 1 - 2|x| & \text{if } |x|+|y| \geq 1 \end{cases}$$

satisfies the appropriate initial condition and majorization, so it suffices to check (3.8). We leave the straightforward verification to the reader and just mention that for the function $A$ and $B$, one can take

$$A(x,y) = \begin{cases} -2x & \text{if } |x|+|y|<1, \\ -2\operatorname{sgn} x & \text{if } |x|+|y| \geq 1, \end{cases} \qquad B(x,y) = \begin{cases} 2y & \text{if } |x|+|y|<1, \\ 0 & \text{if } |x|+|y| \geq 1 \end{cases}$$

(as in the previous example, $A$ and $B$ are essentially the partial derivatives of $U$). This establishes the sharp inequality

$$\lambda \mathbb{P}(|g_n| \geq \lambda) \leq 2\mathbb{E}|f_n|, \qquad n = 0, 1, 2, \ldots,$$

for differentially subordinate martingales.

It is instructive to see that the Steps 1, 2 and 3 have a very transparent probabilistic interpretation. The idea can be described as follows. Let us first look at (3.11). A little informally, we want to make the probability $\mathbb{P}(|g_n| \geq 1)$ as large as possible and the expectation $\mathbb{E}|f_n|$ as little as possible. Furthermore, the process $(|f_n|)_{n\geq 0}$ is a submartingale, so its expectation does not decrease as we increase $n$; however, it stays on the same level if the process does not change its sign.

*Step 1. The case* $|y| \geq 1$. Here the reasoning is simple: the supremum defining $V(x,y)$ must be attained for the constant pair $(f,g) \equiv (x,y)$. Indeed, for such a pair the probability $\mathbb{P}(|g_n| \geq 1)$ is one (so we cannot make it larger); on the other hand, any nontrivial movement of $f$ can only increase $\mathbb{E}|f_n|$. Thus $V(x,y) = 1 - C_1|x|$.

*Step 2. The case* $|x|+|y| \leq 1$. Here it is again possible to send $g$ outside the strip $\mathbb{R} \times (-1,1)$ almost surely and simultaneously keep $\mathbb{E}|f_n|$ at the level $|x|$. Indeed, suppose that $x \geq 0$ and $y \geq 0$ (and $y < 1$), for the remaining cases the construction is similar. We consider the martingale pair $(f,g)$ starting from $(x,y)$ and moving along the line of slope $-1$ at the first step: formally, we require that $(f_1,g_1) \in \{(x+y-1,1),(x+y+1,-1)\}$ (the corresponding probabilities are uniquely determined by the fact that $\mathbb{E}((f_1,g_1)|\mathcal{F}_0) = (x,y)$). The construction is completed by the condition $f_1 = f_2 = f_3 = \ldots$ and $g_1 = g_2 = g_3 = \ldots$ almost surely. Then $\mathbb{P}(|g_1| \geq 1) = 1$ and $\mathbb{E}|f_1| = |x|$, so $V(x,y) = 1 - C_1|x|$.

*Step 3. The case* $|x|+|y| < 1$. Here we need to experiment a bit. A little thought leads to the following idea: start the pair $(f,g)$ at $(x,y)$, then, at the first step, send it to the set $\{(x,y) : x+y \in \{-1,1\}\}$, and then move according to the pattern described in Step 2. Precisely, consider the following Markov martingale $(f,g)$:

(i) It starts from $(x,y)$: $(f_0,g_0) \equiv (x,y)$.
(ii) The random variable $df_1 = dg_1$ is centered and takes values in $\{(1-x-y)/2, (-1-x-y)/2\}$.
(iii) Conditionally on $\{df_1 > 0\}$ and conditionally on $\{df_1 < 0\}$, the random variable $df_2 = -dg_2$ is centered and takes values in $\{-f_1, g_1+1\}$.
(iv) Put $df_n = dg_n \equiv 0$ for $n \geq 3$.

Then we have $\mathbb{P}(|g_2| \geq 1) = 1$. Furthermore, we easily derive that $df_1$ takes values $(1-x-y)/2$ and $(-1-x-y)/2$ with probabilities $p_- = (1+x+y)/2$ and $p_+ = (1-x-y)/2$, respectively. In consequence, since $f_2$ has the same sign as $f_1$, we may write

$$\begin{aligned}\mathbb{E}|f_2| = \mathbb{E}|f_1| &= \left|x + \frac{1-x-y}{2}\right| \cdot \frac{1+x+y}{2} + \left|x + \frac{-1-x-y}{2}\right| \cdot \frac{1-x-y}{2} \\ &= \frac{1+|x|^2-|y|^2}{2}.\end{aligned}$$

and hence we get

$$V(x,y) \geq 1 - C_1(1+|x|^2-|y|^2)/2.$$

The remaining analysis is the same as previously. The above probabilistic analysis very clearly illustrates the notions of the continuation and stopping regions described in Remark (d) above. Indeed, from the formulas for $G$ and $V$ we infer that

$$\begin{aligned} C &= \{(x,y) : V(x,y) > G(x,y)\} = \mathbb{R} \times (-1,1), \\ D &= \{(x,y) : V(x,y) = G(x,y)\} = \mathbb{R} \times ((-\infty,-1] \cup [1,\infty)), \end{aligned}$$

and it is evident from Steps 1, 2 and 3 above that the optimal pairs $(f,g)$ corresponding to $V(x,y)$ have some nontrivial evolution if and only if $(x,y) \in C$.

Example 3.7. Fix $1 < p < 2$. The purpose of the example is to identify the best constant $C_p$ in the $L^p$ estimate

$$\mathbb{E}|g_n|^p \leq C_p^p \mathbb{E}|f_n|^p, \qquad n = 0,\, 1,\, 2,\, \ldots,$$

under the assumption of the differential subordination of $g$ to $f$. As previously, our first step is to consider the case of $\pm 1$-transforms. The inequality is of the form (3.5) with $G(x,y) = |y|^p - C_p^p|x|^p$ and the formula for the smallest Bellman function becomes

$$V(x,y) = \sup\left\{\mathbb{E}|g_n|^p - C_p^p\mathbb{E}|f_n|^p\right\}, \tag{3.12}$$

where the supremum is taken over all $n$ and all pairs $(f,g)$ starting from $(x,y)$ such that $g$ is a $\pm 1$-transform of $f$. As we have explained in Remark (b), we may search for the Bellman function in the class of homogeneous functions:

$$U(\lambda x, \pm\lambda y) = \lambda^p U(x,y), \qquad x,\, y \in \mathbb{R},\, \lambda > 0.$$

Thus it is enough to find the formula for the restriction

$$u(x) = U(x, 1-x). \tag{3.13}$$

This function must be concave and majorize the function $w(x) = G(x, 1-x)$ on $[0,1]$. Clearly, this is not a full set of requirements: we must somehow guarantee that the function $x \mapsto U(x, 1-x)$ is concave on the full real line. To take this into account, let us first inspect the behavior of this restriction on $(1, \infty)$. To this end, fix $x > 1$ and note that

$$U(x,1-x) = U(x, x-1) = (2x-1)^p u\left(\frac{x}{2x-1}\right)$$

and hence, the analysis the one-sided derivatives yields

$$2pu(1) - u'(1-) = \frac{d}{dx}U(x,1-x)\bigg|_{x=1+} \leq \frac{d}{dx}U(x,1-x)\bigg|_{x=1-} = u'(1-),$$

or

$$u'(1-) \geq pu(1). \tag{3.14}$$

A natural guess for $u$ is to take a linear function, whose graph is tangent to that of $w$ at some point $x_0 \in (0,1)$. In other words, as a candidate for $u$, let us take

$$u(x) = w(x_0) + w'(x_0)(x - x_0),$$

for some $x_0$ to be found. An application of (3.14) yields $p\big[w(x_0)+w'(x_0)(1-x_0)\big] \leq w'(x_0)$ or, equivalently,

$$C_p^p \geq \frac{(1-x_0)^{p-2} - (p-1)(1-x_0)^{p-1}}{(p-1)x_0^{p-1}}. \tag{3.15}$$

Now we specify $x_0$ by requiring that the right-hand side above is the least possible, and assume that $C_p^p$ is equal to that minimal value. Simple calculations show that we must take $x_0 = 1 - 1/p$ and $C_p = (p-1)^{-1}$; this leads us to the candidate

$$u(x) = -\frac{p^{3-p}}{p-1}\left(x - 1 + \frac{1}{p}\right)$$

and the function

$$U(x,y) = (|x|+|y|)^p u\left(\frac{|x|}{|x|+|y|}, \frac{|y|}{|x|+|y|}\right) = p^{2-p}\left(|y| - \frac{|x|}{p-1}\right)(|x|+|y|)^{p-1}.$$

Now one has to verify rigorously that the properties 1°, 2° and 3° are satisfied. One can perform this analysis right away in the context of the differential subordination (we omit the details) and thus obtain that

$$\mathbb{E}|g_n|^p \leq (p-1)^{-p}\mathbb{E}|f_n|^p, \qquad n = 0,\ 1,\ 2,\ \ldots.$$

In contrast to the preceding examples, the above analysis does not imply that the constant $(p-1)^{-1}$ is the best possible. To prove this, we will exploit the abstract properties of the function $V$. Suppose that the $L^p$ inequality holds with some constant $C_p$ and introduce the function $v(x) = V(x, 1-x)$, $x \in [0,1]$. We have

$$v((C_p+1)^{-1}) \geq w((C_p+1)^{-1}) = \big(1 - (C_p+1)^{-1}\big)^p - C_p^p(C_p+1)^{-p} = 0. \tag{3.16}$$

Therefore, exploiting the concavity of $v$, we get

$$\frac{C_p+1}{C_p}v(1) \geq \frac{v(1) - v((C_p+1)^{-1})}{1-(C_p+1)^{-1}} \geq v'(1-) \geq pv(1).$$

In the last passage we have used that fact that $v$, being appropriately symmetric and concave, must satisfy (3.14). The above estimate is equivalent to $C_p \geq (p-1)^{-1}$, since $v(1)$ is strictly negative. The latter follows from concavity of $v$, (3.16) and the estimate $v(0) \geq w(0) = 1$.

We should point out here that the function $U$ discovered above does not coincide with the smallest Bellman function $V$. One can show that the two functions coincide only on a part of $\mathbb{R}^2$, more precisely, we have

$$V(x,y)=\begin{cases} U(x,y) & \text{if } |y|\leq |x|/(p-1),\\ |y|^p-(p-1)^{-p}|x|^p & \text{if } |y|>|x|/(p-1).\end{cases}$$

It is instructive to see that the above result can be obtained with the use of a different argumentation which is of probabilistic nature. Let us write down the formula for the function $V$:

$$V(x,y)=\sup\left\{\mathbb{E}|g_n|^p-C_p^p\mathbb{E}|f_n|^p\right\},$$

where the supremum is taken over all $n$ and all martingale pairs $(f,g)$ starting from $(x,y)$ such that $g$ is a $\pm1$ transform of $f$. As in the example concerning the weak-type estimate, we begin by a little informal observation. Namely, when searching for the supremum, we need to make $\mathbb{E}|g_n|^p$ relatively big in comparison to $\mathbb{E}|f_n|^p$. Both processes $(|g_n|^p)_{n\geq0}$ and $(|f_n|^p)_{n\geq0}$ are submartingales, so their moments grow as we increase $n$. However, we can steer the pair $(f,g)$ so that the $p$-th moment of $g$ grows appropriately faster than that of $f$. To do this, note that the second derivative of the function $t\mapsto|t|^p$ is decreasing: this implies that it is "profitable" to evolve the pair $(f,g)$ when $g_n$ "small" and $f_n$ "big"; on the other hand, if $g$ is big when compared to $f$, the best strategy seems to be to stop the processes at once. This observation can be very nicely expressed in terms of the continuation and the stopping regions $C$ and $D$: we should have $C=\{(x,y):|y|<\gamma_p|x|\}$ and $D=\{(x,y):|y|\geq\gamma_p|x|\}$, for some parameter $\gamma_p$ to be found. In other words,

$$V(x,y)=|y|^p-C_p^p|x|^p \qquad \text{if } |y|\geq\gamma_p|x|$$

and $V(x,y)>|y|^p-C_p^p|x|^p$ for $|y|<\gamma_p|x|$. To identify $V$ on $C=\{(x,y):|y|<\gamma_p|x|\}$, we make the second observation. Namely, a little thought suggests that the following principle should hold: if $g$ approaches 0, then $f$ should go away from this value, and vice versa. This means that when $(x,y)\in C\cap\{xy>0\}$, then $V$ should be linear along the lines of slope $-1$, and linear along the lines of slope 1 on the remaining part of $C$. This actually brings us back to the analytic reasoning similar to that presented above. Indeed, one first notes that $V$ is homogeneous of order $p$ and thus it is enough to identify $v(x)=V(x,1-x)$ for $x\in[0,1]$. The above analysis suggests that if $1-x\geq\gamma_p x$ (equivalently: $x\leq(1+\gamma_p)^{-1}$, then

$$v(x)=w(x)=(1-x)^p-C_p^px^p.$$

On the other hand, on $[(1+\gamma_p)^{-1},1]$ the function $v$ should be linear and majorize $w$: this uniquely determines $v$:

$$v(x)=w'((1+\gamma_p)^{-1})(x-(1+\gamma_p)^{-1})+w((1+\gamma_p)^{-1}),$$

and the condition $v'(1-)\geq pv(1)$ implies (3.15), with $x_0=(1+\gamma_p)^{-1}$.

There is a natural question about the explicit examples showing that the constant $(p-1)^{-1}$ cannot be improved. The idea behind the construction is straightforward: we want to start the pair $(f,g)$ at some point $(x,x)$ for which $U(x,x)=0$ and then require that the martingales evolve on the line segments along which $U$ is linear. If, in addition, we ensure that the final pair $(f_N, g_N)$ terminates at the set $D = \{(x,y) : |y| = (p-1)^{-1}|x|\}$, then all the inequalities

$$\mathbb{E}|g_N|^p - C_p^p \mathbb{E}|f_N|^p \leq \mathbb{E}U(f_N, g_N) \leq \mathbb{E}U(f_{N-1}, g_{N-1}) \leq \ldots \mathbb{E}U(f_0, g_0) \leq 0$$

would become equalities, so the constant $C_p^p$ would be attained. Unfortunately, this cannot be done in such a simple manner. The only pair $(f, g)$ which satisfies the above set of conditions is the pair $(f,g) \equiv (0,0)$, and clearly, the equality $\mathbb{E}|g_N|^p = C_p^p \mathbb{E}|f_N|^p$ is not meaningful. We will show that for any $\varepsilon > 0$, there is a martingale pair $(f, g)$ such that if $n$ is sufficiently large, then $\mathbb{E}|g_n|^p > \big((p-1)^{-1} - \varepsilon\big)^p \mathbb{E}|f_n|^p$. To guarantee this inequality, we may relax a little the requirements formulated above. Fix $\beta \in (1, (p-1)^{-1})$. First, we will allow the martingale $(f, g)$ to start from the point $(1, 1)$: we will see that the "loss" $\mathbb{E}U(f_0, g_0) < 0$ we experience here is insignificant in comparison to the overall size of the martingales $f$ and $g$. Next, consider the following Markov transities:

(i) The states lying in the set $\{(x,y) : |y| \geq \beta|x|\}$ are absorbing.

(ii) The state $(x, y)$ with $0 < y < \beta x$, leads to $(x + y, 0)$ or to $\left(\frac{x+y}{\beta+1}, \frac{\beta(x+y)}{\beta+1}\right)$ (the move along the line of slope $-1$).

(iii) The state $(x, 0)$ with $x > 0$ leads to $(x + \delta x, \delta x)$ or to $\left(\frac{x}{\beta+1}, -\frac{\beta x}{\beta+1}\right)$ (the move along the line of slope 1).

(iv) The remaining states $(x, y)$ behave in a symmetrical way when compared to (ii) and (iii).

It is easy to check that $g$ is a $\pm 1$-transform of $f$. Furthermore, $(f, g)$ converges to a nontrivial random variable $(f_\infty, g_\infty)$ with values $\{(x,y) : |y| = \beta|x|\}$. Therefore we will be done if we check that $f$ is $L^p$ bounded. This can be verified readily from the above construction, we leave the details to the reader.

EXAMPLE 3.8. The purpose of our final example is to illustrate the modification of the method when the values of $f$ are restricted to the interval $[-1, 1]$ and the transforming sequence takes values in $[0, 1]$. Namely, in such a setting, we will identify the best constant $C_\lambda$ in the estimate

$$\mathbb{P}(|g_n| \geq \lambda) \leq C_\lambda, \qquad n = 0, 1, 2, \ldots, \tag{3.17}$$

where $\lambda \geq 3/2$. As in the previous examples, we start with the extremal case in which the transforming sequence takes values in $\{0, 1\}$. The inequality (3.17) can be rewritten in the form

$$\mathbb{E}G(f_n, g_n) \leq C_\lambda, \qquad n = 0, 1, 2, \ldots,$$

with $G(x, y) = 1_{\{|y| \geq \lambda\}}$, $(x, y) \in [-1, 1] \times \mathbb{R}$. This inequality is of slightly different form than (3.1), due to the appearance of the term $C_\lambda$ on the right. Of course, we could have put it on the left and use the function $G(x, y) = 1_{\{|y| \geq \lambda\}} - C_\lambda$, but instead we prefer to say that the method described above works perfectly fine, the only change we need is the requirement $U(x, y) \leq C_\lambda$ in the initial condition 1°.

Let us start with the smallest Bellman function

$$V(x, y) = \sup \mathbb{E}G(f_n, g_n) = \sup \Big\{ \mathbb{P}(|g_n| \geq \lambda) - C_\lambda \Big\},$$

where the supremum is taken over all associated parameters. The function $G$ is symmetric with respect to $x$ and $y$; this implies that we have $V(x, y) = V(-x, -y)$ for all $(x, y) \in [-1, 1] \times \mathbb{R}$ (on contrary, we do *not* have $V(x, -y) = V(x, y)$: this is due to the fact that the transforming sequence is non-symmetric, i.e., takes values in a non-symmetric set).

*Step 1.* As in the preceding examples, it is easy to identify $V$ on some part of its domain, directly from the definition. It will be convenient to express the observations in terms of the continuation and stopping sets $C$ and $D$. Directly from the above definition of $V$, we see that our objective is to send $g$ outside $(-1,1)$ with as large probability as possible, keeping $f$ inside the interval $[-1,1]$. This observation immediately gives $C=(-1,1)\times(-\lambda,\lambda)$ and $D=([-1,1]\times\mathbb{R})\setminus C$. Indeed, we have $G=0$ on $(-1,1)\times(-\lambda,\lambda)$ and for any $(x,y)\in(-1,1)\times(-\lambda,\lambda)$ one easily constructs a martingale pair $(f,g)$ starting from $(x,y)$ for which the probability $\mathbb{P}(|g_n|\geq\lambda)$ is strictly positive for some $n$. On the other hand, for any $(x,y)\notin(-1,1)\times(-\lambda,\lambda)$, the best choice is to take the constant martingale $(f,g)\equiv(x,y)$. Indeed, if $|x|=1$ this is due to the fact that $f$ must be stopped immediately (otherwise it leaves $[-1,1]$), while for remaining points the process $|g|$ already reaches the desired set $[\lambda,\infty)$ at its initial position. This reasoning shows that $V(x,y)=1_{\{|y|\geq\lambda\}}$ on $D$ and it remains to identify the formula for $V$ on $C$. We will actually guess this formula basing on a number of (reasonable) assumptions: this is done in the next four steps and our reasoning will be a little informal. The rigorous verification that the obtained candidate enjoys all the required properties is a separate issue (see Step 6). To stress that we are working with a candidate, we will use a different letter and, from now on, denote the investigated function by $U$.

*Step 2.* In all the considerations below, we will treat $U$ as a $C^1$ function (though the candidate we will end up with will not even be continuous). Let us consider the case when $x-1+\lambda\leq y<\lambda$. Consider the line of slope 1 passing through $(x,y)$. The restriction of $G$ to this line (more formally, to an appropriate line segment) is given by

$$\xi(t)=\begin{cases}0 & \text{if } t\in[-1-x,\lambda-y),\\ 1 & \text{if } t\in[\lambda-y,1-x],\end{cases}$$

so the least concave majorant is $\zeta(t)=\min\{(t+x+1)/(x-y+\lambda+1),1\}$. Consequently, we have

$$V(x,y)\geq\frac{x+1}{x-y+\lambda+1}.$$

The right hand side is linear along lines of slope 1 and concave along lines of slope 0 when $y<\lambda$. These desired properties suggest to assume that

$$U(x,y)=\frac{x+1}{x-y+\lambda+1}\qquad\text{if } x-1+\lambda\leq y<\lambda$$

(with a symmetric formula for $-x-1+\lambda\leq -y<\lambda$).

*Step 3.* Let us turn our attention to the case $y<x-1+\lambda$. Suppose, for a while, that $y$ is positive and not too close to 0. A little thought and experimentation suggest the following behavior of $U$: there should be some curve $\gamma$ splitting the set $\{(x,y)\in[-1,1]\times\mathbb{R}:y<x-1+\lambda\}$ such that $U$ is linear along the lines of slope 1 on the left of $\gamma$, and linear along the horizontal lines on the right of $\gamma$. Let us parametrize this curve as $\{(\gamma(y),y):y\in I\}$ for some interval $I$. So, if $A$ denotes the restriction of $U$ to the curve $\{(\gamma(y),y):y\in I\}$ (meaning that $A(y)=U(\gamma(y),y)$)

and $x > \gamma(y)$, then

$$U(x,y) = \frac{x-\gamma(y)}{1-\gamma(y)}V(1,y) + \frac{1-x}{1-\gamma(y)}A(y) = \frac{1-x}{1-\gamma(y)}A(y). \tag{3.18}$$

Let us assume that the function $U$ is of class $C^1$ on the set $y < x-1+\lambda$ (actually, this condition will not hold, but it leads to the right candidate). Then the aforementioned linearity of $U$ along line segments of slope $-1$ lying on the left of $\gamma$ implies that

$$U_x(\gamma(y),y) + U_y(\gamma(y),y) = \frac{U(\gamma(y),y) - U(-1, y-\gamma(y)-1)}{\gamma(y)+1} = \frac{A(y)}{\gamma(y)+1}.$$

The partial derivatives can be identified with the use of (3.18): as the result, we obtain the differential equation

$$\frac{A'(y)}{A(y)} + \frac{\gamma'(y)}{1-\gamma(y)} = \frac{2}{1-\gamma^2(y)}.$$

Solving this equation, we get

$$\frac{A(y)}{1-\gamma(y)} = c\exp\left(\int_0^y \frac{2\mathrm{d}u}{1-\gamma^2(u)}\right),$$

for some unknown constant $c$. To find this parameter, let $(\gamma(y_0), y_0)$ be the intersection of the curve $\gamma$ with the line $y = x-1+\lambda$ (the "upper end" of $\gamma$). We have assumed that $U$ is continuous, so by the formula guessed at the previous step we may write

$$c\exp\left(\int_0^{y_0} \frac{2\mathrm{d}u}{1-\gamma^2(u)}\right) = \frac{A(y_0)}{1-\gamma(y_0)} = \frac{1+\gamma(y_0)}{2(1-\gamma(y_0))}.$$

Consequently, we have obtained that

$$\frac{A(y)}{1-\gamma(y)} = \frac{1+\gamma(y_0)}{2(1-\gamma(y_0))}\exp\left(\int_{y_0}^y \frac{2\mathrm{d}u}{1-\gamma^2(u)}\right).$$

Now suppose that $y$ is a positive number and suppose that $x > \gamma(y)$. Exploiting the linearity of $U$ along the horizontal segments, we may write

$$\begin{aligned} U(x,y) &= \frac{1-x}{1-\gamma(y)}A(y) + \frac{x-\gamma(y)}{1-\gamma(y)}U(1,y) \\ &= \frac{1+\gamma(y_0)}{2(1-\gamma(y_0))}(1-x)\exp\left(\int_{y_0}^y \frac{2\mathrm{d}u}{1-\gamma^2(u)}\right). \end{aligned} \tag{3.19}$$

At this point it is not difficult to see what the optimal choice for $\gamma$ should be. The function $\gamma$ should be continuous, take values in $(-1,1)$ and satisfy $\gamma(y) \geq y+1-\lambda$ (the latter condition means that the curve $\gamma$ lies below or on the line $y = x-1+\lambda$). Now if we keep $y_0$ and $\gamma(y_0)$ fixed, it is clear that in order to maximize $U(x,y)$, we need to make $\gamma$ as close to 0 as possible: this will guarantee that the integral $\int_{y_0}^y 2(1-\gamma^2(u))^{-1}\mathrm{d}u$ will be maximal (recall that $y \leq y_0$). This leads to the assumption $\gamma \equiv 0$ and $y_0 = \lambda - 1$. Having assumed this, we can find $U$ on a large part of the domain. Namely, if $\lambda - 1 \leq y < x+\lambda-1$, then

$$U(x,y) = \frac{1-x}{\lambda-y}U(y+1-\lambda,y) + \frac{x-y-1+\lambda}{\lambda-y}U(1,y) = \frac{(1-x)(y-\lambda+2)}{2(\lambda-y)}.$$

If $x \in [0,1]$ and $y < \lambda - 1$ is sufficiently big (this will be made more precise later), then the above considerations (see (3.19)) give

$$U(x,y) = \frac{1-x}{2}\exp(2(y-\lambda+1)). \tag{3.20}$$

Finally, if $x \in [-1,0]$ and $y < x + \lambda - 1$ is large enough, we have

$$\begin{aligned} U(x,y) &= -xU(-1,-x+y-1) + (1+x)U(0,y-x) \\ &= (1+x)A(-x+y) \\ &= \frac{1+x}{2}\exp(2(-x+y-\lambda+1)). \end{aligned} \tag{3.21}$$

*Step 4.* Now it is time to specify what we have meant by saying that the formulas (3.20) and (3.21) hold for sufficiently large $y$. Clearly, they cannot hold for all $y \leq \lambda - 1$, since then the symmetry condition $U(x,y) = U(-x,-y)$ would be violated. A closer look at this symmetry condition suggests that (3.20) should hold true for $D_1 = \{(x,y) : x \in [0,1],\ 1/2 \leq y \leq \lambda - 1\}$ and (3.21) should be valid for $D_2 = \{(x,y) : x \in [-1,0], x - 1/2 \leq y \leq x + 1 - \lambda\}$: indeed, the lower boundaries $y = 1/2$ in $D_1$ and $y = x - 1/2$ are the smallest numbers with the property that the interiors of the sets $D_1$, $D_2$ and their reflections $-D_1 = \{(x,y) : (-x,-y) \in D_1\}$, $-D_2$ are disjoint.

*Step 5.* It remains to find $U$ on the set $\{(x,y) : y - 1/2 \leq x \leq y + 1/2,\ y \in [-1,1]\}$. Now we make the following guess. Namely, take the line segment of slope 1 passing through $(x,y)$, with endpoints lying on the lines $y = \pm 1/2$. We assume that $U$ is linear on this line segment, which leads us to

$$\begin{aligned} U(x,y) &= \left(\frac{1}{2} - y\right) U\left(x - y - \frac{1}{2}, -\frac{1}{2}\right) + \left(y + \frac{1}{2}\right) U\left(x - y + \frac{1}{2}, \frac{1}{2}\right) \\ &= \exp(3 - 2\lambda)\left(y^2 - xy + \frac{1}{4}\right). \end{aligned}$$

Summarizing, we have obtained the function uniquely determined by the condition $U(x,y) = U(-x,-y)$ and the formula

$$U(x,y) = \begin{cases} 1 & \text{if } y \geq \lambda, \\ \dfrac{x+1}{x-y+\lambda+1} & \text{if } x - 1 + \lambda \leq y < \lambda, \\ \dfrac{(1-x)(y-\lambda+2)}{2(\lambda-y)} & \text{if } \lambda - 1 \leq y < x + \lambda - 1, \\ \dfrac{1-x}{2}\exp(2(y-\lambda+1)) & \text{if } x \in [0,1],\ 1/2 \leq y < \lambda - 1, \\ \dfrac{1+x}{2}\exp(2(-x+y-\lambda+1)) & \text{if } x \in [-1,0],\ -1/2 \leq y - x < \lambda - 1, \\ \exp(3-2\lambda)\left(y^2 - xy + \dfrac{1}{4}\right) & \text{if } -1/2 \leq x - y \leq 1/2,\ y \in [-1,1]. \end{cases}$$

The initial condition 1° reads

$$U(x,y) \leq C_\lambda \qquad \text{if } x \in [-1,1] \text{ and } y \in \{0, x\},$$

so in particular, taking $x = y = 0$, we get $C_\lambda \geq \exp(3-2\lambda)/4$. We *assume* that we have equality here.

*Step 6.* The above $U$ is just a candidate for the Bellman function. Furthermore, it was constructed under the assumption that the transforming sequence takes values in $\{0,1\}$. The remaining part of the analysis is to verify the conditions

1° We have $U(x,y) \leq \exp(3-2\lambda)/4$ for all $x \in [-1,1]$ and $y \in \{0,x\}$.

2° We have $U(x,y) \geq 1_{\{|y|\geq 1\}}$ for all $(x,y) \in [-1,1] \times \mathbb{R}$.

3° The function $U$ is concave along any line segment of slope belonging to $[0,1]$, contained in $[-1,1] \times \mathbb{R}$.

We leave the lengthy, but rather straightforward verification to the reader. Having done this, we will have shown the estimate

$$\mathbb{P}(|g_n| \geq \lambda) \leq \exp(3-2\lambda)/4$$

under the assumption that $f$ is bounded by 1 and $g$ is its full transform by a predictable sequence with values in $[0,1]$.

*Step 7.* Finally, let us address the issue of the sharpness of the constant $C_\lambda$. It is not difficult to see the structure of the (almost) extremal examples. Informally speaking, the corresponding pair $(f,g)$ should start from some point $(x,x)$ for which we have $U(x,x) = C_\lambda$, and then it should move along the segments of linearity of $U$. This will guarantee that in the chain of inequalities

$$\mathbb{E}U(f_n,g_n) \leq \mathbb{E}U(f_{n-1},g_{n-1}) \leq \ldots \mathbb{E}U(f_0,g_0) \leq C_\lambda,$$

we have actually a chain of equalities (or almost equalities). If in addition we ensure that $(f,g)$ terminates at the stopping set $D$, we will be able to write that for a large $n$ we have $\mathbb{P}(|g_n| \geq 1) = \mathbb{E}G(f_n,g_n) = C_\lambda$ (or rather $\approx C_\lambda$) and we will be done.

Let us put the above idea into a rigorous framework: for simplicity, we will consider the strict inequality $\lambda > 3/2$ only. We assume that $(f,g)$ starts from the point $(1/2,1/2)$ and, at the first step, jumps either to $(0,1/2)$ or to $(1,1/2)$ (with probabilities $1/2$). The latter point belongs to the stopping set $\{(x,y) : U(x,y) = G(x,y)\}$, so we finish the evolution there (we can also argue that the martingale $f$ must stop, since otherwise it would leave the interval $[-1,1]$). On contrary, if $(f_1,g_1) = (0,1/2)$, then the movement continues. Unfortunately, there is no line segment of slope 0 or 1, containing $(0,1/2)$ inside, along which $U$ is linear. To overcome this difficulty, we fix a small positive $\delta$ and move $(f,g)$ along the line segment with endpoints $(-1,-1/2)$ and $(\delta, 1/2+\delta)$: we expect that the "loss" obtained due to this non-optimal move (i.e., the difference $\mathbb{E}(U(f_2,g_2)|(f_1,g_1) = (0,1/2)) - U(0,1/2))$ will be of order $o(\delta)$ and hence insignificant if we let $\delta \to 0$ at the very end. Formally, on the set $\{(f_1,g_1) = (0,1/2)\}$, we assume that the random variable $(f_2,g_2)$ takes values in the set $\{(-1,-1/2),(\delta,1/2+\delta)\}$ (the corresponding probabilities $\delta/(1+\delta)$ and $1/(1+\delta)$ are determined by the requirement that $(f,g)$ is a martingale). If $(f_2,g_2) = (-1,-1/2)$, the movement stops; if $(f_2,g_2) = (\delta,1+\delta)$, we let $(f_3,g_3)$ jump to $(0,1+\delta)$ or to $(1,1+\delta)$. If the latter occurs, the evolution is over; if $(f_3,g_3) = (0,1+\delta)$, we move $(f_4,g_4)$ along the line segment with endpoints $(-1,-1/2+\delta)$ and $(\delta,1+2\delta)$, and so on.

Now suppose that $\delta$ is of special form: $\delta = (\lambda - 3/2)/N$ for some large integer $N$. It is easy to see that after $2N+1$ steps we have two possibilities: either $f_{2N+1} = \pm 1$ (and we have stopped the evolution of the martingale pair), or $(f_{2N+1},g_{2N+1}) = (0,\lambda-1)$. If the latter occurs, we change the above scheme and let $(f_{2N},g_{2N})$ move along the line segment of slope 1, with endpoints $(-1,\lambda-2)$ and $(1,\lambda)$. Then, after this $2N+2$-nd step, we ultimately stop the process.

It is clear that the martingale $f$ just constructed takes its values in $[-1,1]$ and $g$ is its full transform by the sequence with values in $\{0,1\}$. Furthermore, we have

$$\begin{aligned}\mathbb{P}(g_{2N+2}\geq\lambda)&=\mathbb{P}(f_0=1/2,f_1=0,f_2=\delta,f_3=0,\ldots,f_{2N+1}=0,f_{2N+2}=1)\\&=\frac{1}{2}\cdot\frac{1}{1+\delta}\cdot(1-\delta)\cdot\frac{1}{1+\delta}\cdot(1-\delta)\cdot\ldots\cdot(1-\delta)\cdot\frac{1}{2}\\&=\frac{1}{4}\left(\frac{1-\delta}{1+\delta}\right)^N.\end{aligned}$$

If we recall that $\delta=(\lambda-3/2)/N$ and let $N\to\infty$, we see that the above quantity converges to $C_\lambda$. This proves the sharpness of the estimate and completes the analysis.

## 3. Problems

**1.** Find the smallest Bellman function corresponding to the inequality

$$\mathbb{P}(g_n\geq 1)\leq 2\mathbb{E}|f_n|,\qquad n=0,\,1,\,2,\,\ldots.$$

**2.** Let $1\leq p<\infty$. Prove that the best constant in the weak-type inequality

$$\mathbb{P}(|g_n|\geq 1)\leq C_p^p\mathbb{E}|f_n|^p,\qquad n=0,\,1,\,2,\,\ldots,$$

is given by

$$C_p^p=\begin{cases}\dfrac{2}{\Gamma(p+1)} & \text{if } 1\leq p\leq 2,\\ \dfrac{p^{p-1}}{2} & \text{if } p>2.\end{cases}$$

**3.** Let $a<b$ are fixed real numbers. Find the best constant $C$ in the weak-type inequality

$$\mathbb{P}(|g_n|\geq 1)\leq C\mathbb{E}|f_n|,\qquad n=0,\,1,\,2,\,\ldots,$$

under the assumption that $g$ is a transform of $f$ by a predictable sequence with values in $[a,b]$.

**4.** For any $K>1$, find the least constant $L(K)$ in the logarithmic estimate

$$\mathbb{E}|g_n|\leq K\mathbb{E}|f_n|\log|f_n|+L(K),\qquad n=0,\,1,\,2,\,\ldots.$$

**5.** For any $1<p<\infty$, find the least constant $C_p$ in the inequality

$$\mathbb{E}|g_n|\leq C_p\left(\mathbb{E}|f_n|^p\right)^{1/p},\qquad n=0,\,1,\,2,\,\ldots,$$

where $f$, $g$ are martingales such that $g$ is differentially subordinate to $f$.

**6.** Find the best constants in the estimates

$$\mathbb{P}(|X_t|+|Y_t|\geq 1)\leq C_1\mathbb{E}|X_t|,\qquad t\geq 0,$$

$$\mathbb{P}(X_t^2+Y_t^2\geq 1)\leq C_2\mathbb{E}X_t^2,\qquad t\geq 0,$$

under the assumption that $X$, $Y$ are orthogonal martingales such that $Y$ is differentially subordinate to $X$.

**7.** Find the best constant $C$ in the inequality

$$\mathbb{P}(X_t^2+Y_t^2\geq 1)\leq C\mathbb{E}|X_t|,\qquad t\geq 0,$$

under the assumption that $X$, $Y$ are orthogonal martingales such that $Y$ is differentially subordinate to $X$.

**8.** For a given $K > 0$, find the least $L = L(K)$ such that if $X$, $Y$ are orthogonal martingales, $X$ is nonnegative and $Y$ is differentially subordinate to $X$, then

$$\mathbb{E}|Y_t| \leq K\mathbb{E}X_t \log X_t + L(K), \qquad t \geq 0.$$

**9.** Let $X$, $Y$ be orthogonal martingales such that $Y$ is differentially subordinate to $X$. Prove that the inequality

$$\mathbb{P}(Y_t \geq 1) \leq \mathbb{E}|X_t|^2, \qquad t \geq 0,$$

is sharp. Find the least Bellman function corresponding to this estimate.

**10.** Let $X$, $Y$ be orthogonal martingales such that $Y$ is differentially subordinate to $X$. Prove that for any $t \geq 0$ we have the sharp inequality

$$\mathbb{P}(Y_t \geq 1) \leq \mathbb{E}|X_t|.$$

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