1 Elements of Category Theory.

Definition 1. A diagram $F : \mathbb{N} \longrightarrow \mathcal{C}$ is called a telescope. A diagram $G : \mathbb{N}^{op} \longrightarrow \mathcal{C}$ is called a tower.

1.1. • Let

$$X_1 \xrightarrow{f_1} X_2 \xrightarrow{f_2} \dots \qquad \xrightarrow{f_{n-1}} X_n \xrightarrow{f_n} \dots$$

be a telescope diagram in $\mathcal{C}$. Assume that $\mathcal{C}$ has direct products and push-outs. We define the shift map $s : \coprod_{n \geq 0} X_n \to \coprod_{n \geq 1} X_n$ by the formula $s = (i_2 \circ f_1, i_3 \circ f_2, \dots, i_{n+1} \circ f_n, \dots)$, where $i_n : X_n \longrightarrow \coprod_{n \geq 0} X_n$ is a canonical map. Show that in the pushout square

$$\begin{array}{ccc} \coprod_{n\geq 0} X_n \sqcup \coprod_{n\geq 0} X_n & \xrightarrow{(s,id)} & \coprod_{n\geq 0} X_n \\ \downarrow & & \downarrow \\ \coprod_{n\geq 0} X_n & \longrightarrow & P \end{array}$$

the pushout P is a colimit for the telescope. Formulate and prove the dual result for towers and limits.

1.2. Show that if in $\mathcal{C}$ there exist equalizers of any pair of morphisms and there exist products of any family of objects, then in $\mathcal{C}$ there exist limits indexed by any category. Formulate and prove the dual result for colimits.

1.3. Functors $L : \mathcal{C} \longrightarrow \mathcal{D}$ and $R : \mathcal{D} \longrightarrow \mathcal{C}$ are the pair of adjoint functors (R is right adjoint), then R commutes with limits(i.e. inverse limits) and L commutes with colimits (i.e. direct limits) .

1.4. • Prove that for a pointed spaces $(X.x_0)$, (Y, y_0) ,

$$map_*((X, x_0), \Omega(Y, y_0)) \cong map_*(S(X, x_0), (Y, y_0)),$$

where S denotes the reduced suspension and $\cong$ a homeomorphism. Show that

$$[(X, x_0), \Omega(Y, y_0)]_* = [S(X, x_0), (Y, y_0)]_*$$

1.5. Show that for $0 \geq k \geq n$, $map_*(S^n, X) \cong \Omega^k(map_*(S^{n-k}, X))$.

2 Fibrations and Cofibrations

2.1. • Consider the commutative square

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle h} \\ C & \xrightarrow{i} & D \end{array} \quad .$$

Prove:

(a) If the square is a pushout square, g is a cofibration, and f is a homotopy equivalence, then i is also a homotopy equivalence.

(b) If the square is a pullback square, h is a fibration, and i is a homotopy equivalence, then f is also a homotopy equivalence.

(c) Give examples, that the assumptions on g being a cofibration and f being a fibration are crucial.

2.2. Find the homotopy fibre of $\mathbb{C}P^1 \hookrightarrow \mathbb{C}P^\infty$ and of $\mathbb{R}P^1 \hookrightarrow \mathbb{R}P^\infty$.

2.3. Show that the homotopy fiber of $X \vee X \to X$ is $\Sigma\Omega X$.

(*Technical step*: Let $f : P \to X$ be a map from a contractible space P. Then the push-out of mapping cylinders

$$\begin{array}{ccc} P & \longrightarrow & Z(f) \\ \downarrow & & \\ Z(f) & & \end{array}$$

is homotopy equivalent to $X \vee X$.)

2.4. • Describe the natural equivalence of functors $\mathcal{M}or(\mathcal{T})_* \longrightarrow \mathcal{T}_*$:

$$C(Sf) \approx S(Cf) \qquad P(\Omega(f) \approx \Omega(P(f)).$$

2.5. For a map $f : X \longrightarrow Y$ describe the natural map $g : \Sigma P(f) \longrightarrow C(f)$ and an adjoint map $g^\# : P(f) \longrightarrow \Omega C(f)$.

3 Homotopy groups.

3.1. • Compute homotopy groups π_i of the Stiefel manifold $V_k(\mathbb{R}^n)$ for $i \leq n{-}k$. (Assumption: $n > 1$.)

3.2. • Suppose B is path connected and $p : E \longrightarrow B$ a fibration with fiber F. Show, that if F is $n-$simple, then there exists a well defined homomorphism:

$$\pi_1(B, b_0) \longrightarrow Aut(\pi_n(F)).$$

3.3. • Show that the long exact sequence of homotopy groups of the pair (X, A), X and A path connected, is a sequence of $\pi_1(A, a_0)$ modules.

3.4. • Show that if the fibration has a section, then $\pi_n(E) \simeq \pi_n(B) \oplus \pi_n(F)$. (Assumption: F connected, $n > 1$) Show that if the fiber contracts to a point in E, then $\pi_n(B) \simeq \pi_n(E) \oplus \pi_{n-1}(F)$. (Assumption: $n > 1$.)

4 CW complexes

4.1. • Suppose X, Y are finite CW-complexes. Describe the structure of a CW- complex on a cartesian product $X \times Y$. Why the finiteness assumption? .Describe the cell structure on $S^k \times S^l$ consisting of four cells.

4.2. For finite CW complexes X, Y describe the structure of a CW- complex on $X \vee Y$.

4.3. • Show that if $p : \tilde{X} \longrightarrow X$ os a covering and X is a CW-complex, then $\tilde{X}$ poseses a natural CW structure for which the map p is cellular.

4.4. Show that if X is a $(n-1)$ connected CW complex of dimension $< 2n$, then there is a structure of an co-H space on X. if X is a $(n-1)$ connected CW complex and $\pi_k(X) = 0$ for $k \geq 2n-1$, then there is a structure of an H space on X.

4.5. Suppose that X is a CW complex of dimension $\leq 2n+2$ for some $n \geq 0$. Prove that the diagonal map $d : X \to X \times X$ is homotopic to a map d' such that $d'(X) \subset (X \times X^n) \cup (X^n \times X)$, where X^n is the n-th skeleton of X.

4.6. If X is a CW-complex of type $K(\pi, n)$ for $n > 1$ and Y is an arbitrary CW-complex, then $\pi_n(X \vee Y) \simeq \pi_n(Y) \oplus \bigoplus_{\lambda \in \pi_1(Y)} \pi_\lambda$ where $\pi_\lambda = \pi$ for each λ.

4.7. • Given a sequence of groups $\{\pi_q\}_{q>0}$ with π_q abelian for $q > 1$, and given an action of π_1 as a group of operators on each π_q for $q > 1$, prove that there is a space Y which realizes this sequence (that is $\pi_q(Y) \simeq \pi_q$ and $\pi_l(Y)$-actinon on $\pi_q(Y)$ corresponds to the action of π_1 on π_q).

4.8. Suppose (X, x_0) is a connected pointed CW-complex, and (Y, y_0) a pointed path connected space. Let $n > 0$ be such that $\pi_i(X, x_0) = 0$ for $i < n$ and $\pi_i(Y, y_0) = 0$ for $i > n$. Show that the assignment $[X, Y]_* \ni f \rightsquigarrow$ $f_\#\ Hom(\pi_n(X, x_0), \pi_n(Y, y_0))$ is a bijection. Show then that $[X, Y]\ Hom(\pi_n(X, x_0), \pi_n(Y, y_0))/Inn(\pi_n(Y, y_0))$ $Rep(\pi_n(X, x_0), \pi_n(Y, y_0))$.

5 Whitehead theorem

5.1. Prove that $\pi_n(S^2) = \pi_n(S^3 \times CP(\infty))$ for every n. Prove, that S^2 is not homotopy equivalent to $S^3 \times CP(\infty)$. Why this does not contradict Whitehead theorem?

5.2. • Prove Whitehead theorem using HELP (i.e. in May). Compare the proof with the one given during the lecture.