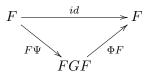
# Algebraic Topology I. – homework problems.

#### 9 listopada 2014

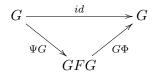
## Series 1: Categories and functors

**Zad. 1.** Show that if a functor  $F: \mathcal{C} \to \mathcal{D}$  has a left (resp. right) adjoint functor then this adjoint functor is unique up to natural equivalence.

**Zad. 2.** Let  $F: \mathcal{C} \to \mathcal{D}$  and  $G: \mathcal{D} \to \mathcal{C}$  is a pair of adjoint functors. Show, that there exist natural transformations  $\Phi: FG \to id_{\mathcal{D}}$  and  $\Psi: id_{\mathcal{C}} \to GF$  such that the triangles of natural transformations:



and



commute.

Let  $F: \mathcal{C} \to \mathcal{D}$  and  $G: \mathcal{D} \to \mathcal{C}$  be a pair of functors such that there exist natural transformations  $\Phi: FG \to id_{\mathcal{D}}$  and  $\Psi: id_{\mathcal{C}} \to GF$  such that the above triangles of natural transformations commute. Show that F, G is a pair of adjoint functors.

**Zad. 3.** Show that if  $F: \mathcal{C} \to \mathcal{D}$  is an equivalence of categories then the functor  $G: \mathcal{D} \to \mathcal{C}$  establishing this equivalence is both right and left adjoint to F. Is the converse true?

## Series 2: Representability, limits.

**Zad. 4.** Let  $F: \mathcal{C} \to \mathcal{D}$  be a functor. For an object  $Y \in ob\mathcal{D}$  consider the functor  $F_Y: \mathcal{C}^{op} \to \mathcal{S}et$ ,  $F_Y(X) = Mor_{\mathcal{D}}(F(X), Y)$ . Prove that if for every object  $Y \in ob\mathcal{D}$  functor  $F_Y$  is representable then there exists a functor  $G: \mathcal{D} \to \mathcal{C}$  right adjoint to F.

**Zad. 5.** Let  $\mathcal{I}$  be a small category and  $F: \mathcal{I} \to \mathcal{C}$  a diagram in  $\mathcal{C}$ . For every object  $X \in ob\mathcal{C}$  define a constant functor  $\Delta_X: I \to \mathcal{C}$ , which to every object  $i \in ob\mathcal{I}$  assigns X and to every morphism in  $\mathcal{I}$  assigns  $id_X$ .  $\Delta: \mathcal{C} \to \mathcal{F}unct(\mathcal{I}, \mathcal{C})$  for which  $\Delta(X) = \Delta_X$  and for a morphism  $f: X \to X'$  in  $\mathcal{C}$  is a

natural transformation  $\Delta(f): \Delta_X \to \Delta_{X'}, \ \Delta(f)(i) = f: \Delta_X(i) = X \to \Delta_{X'(i)} = X'$ , Check that  $\Delta$  is indeed a functor. Prove that  $\lim_{I \to \infty} F$  exists iff the functor  $Mor_{\mathcal{F}unct(\mathcal{I},\mathcal{C})}(\Delta -, F): \mathcal{C}^{op} \to \mathcal{S}et$  is representable and  $\lim_{I \to \infty} F$  is the representing object.

Formulate the analogous statement for colim.

**Zad. 6.** Let  $\mathcal{I}$  be a small category and consider a diagram  $F: \mathcal{I} \to \mathcal{S}et$  in the category of sets given by a representable functor  $Mor_{\mathcal{I}}(i_0, \cdot)$ . Find colim F. (hint: Yoneda helps a lot!)

### 1 Series 3: Cofibrations and Fibrations.

Zad. 7. Show that if

$$\begin{array}{ccc}
A & \xrightarrow{j} & X \\
f \downarrow & & \downarrow h \\
Y & \xrightarrow{g} & W
\end{array}$$

is a push out diagram in Top then for every space Z the induced diagram

$$\begin{array}{c|c} map(W,Z) & \xrightarrow{g^*} & map(Y,Z) \\ & h^* \downarrow & & \downarrow f^* \\ map(X,Z) & \xrightarrow{j^*} & map(A,Z) \end{array}$$

is a pull back diagram. (This was the key step in proving that for a map  $f: X \to Y$ ,  $map(Z(f), Z) = P(f^*)$ .)

**Zad. 8.** Present the map  $X \coprod X \to X$  sending each summand identically onto X as the composition of a cofibration and homotopy equivalence.

Present the diagonal map  $\Delta: Y \to Y \times Y$  as the composition of a homotopy equivalence and a fibration.

**Zad. 9.** Present the map  $X \to \{*\}$  as the composition of a cofibration and homotopy equivalence. Present the map  $\{*\} \to X$  as the composition of a homotopy equivalence and a fibration.