# Algebraic Topology I. – homework problems.

9 listopada 2014

## Series 1: Categories and functors

**Zad. 1.** Show that if a functor $F : \mathcal{C} \to \mathcal{D}$ has a left (resp. right) adjoint functor then this adjoint functor is unique up to natural equivalence.

**Zad. 2.** Let $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{C}$ is a pair of adjoint functors. Show, that there exist natural transformations $\Phi \colon FG \to id_{\mathcal{D}}$ and $\Psi \colon id_{\mathcal{C}} \to GF$ such that the triangles of natural transformations:

$$\begin{tikzcd} F \arrow[rr, "id"] \arrow[dr, "F\Psi"'] & & F \\ & FGF \arrow[ur, "\Phi F"'] & \end{tikzcd}$$

and

$$\begin{tikzcd} G \arrow[rr, "id"] \arrow[dr, "\Psi G"'] & & G \\ & GFG \arrow[ur, "G\Phi"'] & \end{tikzcd}$$

commute.
Let $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{C}$ be a pair of functors such that there exist natural transformations $\Phi \colon FG \to id_{\mathcal{D}}$ and $\Psi \colon id_{\mathcal{C}} \to GF$ such that the above triangles of natural transformations commute. Show that $F$, $G$ is a pair of adjoint functors.

**Zad. 3.** Show that if $F : \mathcal{C} \to \mathcal{D}$ is an equivalence of categories then the functor $G : \mathcal{D} \to \mathcal{C}$ establishing this equivalence is both right and left adjoint to $F$. Is the converse true?

## Series 2: Representability, limits.

**Zad. 4.** Let $F : \mathcal{C} \to \mathcal{D}$ be a functor. For an object $Y \in ob\mathcal{D}$ consider the functor $F_Y : \mathcal{C}^{op} \to \mathcal{S}et$, $F_Y(X) = Mor_{\mathcal{D}}(F(X), Y)$. Prove that if for every object $Y \in ob\mathcal{D}$ functor $F_Y$ is representable then there exists a functor $G : \mathcal{D} \to \mathcal{C}$ right adjoint to $F$.

**Zad. 5.** Let $\mathcal{I}$ be a small category and $F : \mathcal{I} \to \mathcal{C}$ a diagram in $\mathcal{C}$. For every object $X \in ob\mathcal{C}$ define a constant functor $\Delta_X : I \to \mathcal{C}$, which to every object $i \in ob\mathcal{I}$ assigns $X$ and to every morphism in $\mathcal{I}$ assigns $id_X$. $\Delta : \mathcal{C} \to \mathcal{F}unct(\mathcal{I}, \mathcal{C})$ for which $\Delta(X) = \Delta_X$ and for a morphism $f : X \to X'$ in $\mathcal{C}$ is a natural transformation $\Delta(f) : \Delta_X \to \Delta_{X'}$, $\Delta(f)(i) = f : \Delta_X(i) = X \to \Delta_{X'(i)} = X'$, Check that $\Delta$ is indeed a functor. Prove that $lim_I F$ exists iff the functor $Mor_{\mathcal{F}unct(\mathcal{I},\mathcal{C})}(\Delta -, F) : \mathcal{C}^{op} \to \mathcal{S}et$ is representable and $lim_F$ is the representing object.

Formulate the analogous statement for colim.

**Zad. 6.** Let $\mathcal{I}$ be a small category and consider a diagram $F : \mathcal{I} \to \mathcal{S}et$ in the category of sets given by a representable functor $Mor_{\mathcal{I}}(i_0, \cdot)$. Find $colimF$. (hint: Yoneda helps a lot!)

# 1 Series 3: Cofibrations and Fibrations.

**Zad. 7.** Show that if

$$\begin{tikzcd} A \arrow[r, "j"] \arrow[d, "f"'] & X \arrow[d, "h"] \\ Y \arrow[r, "g"] & W \end{tikzcd}$$

is a push out diagram in $\mathcal{T}op$ then for every space $Z$ the induced diagram

$$\begin{tikzcd} map(W,Z) \arrow[r, "g^*"] \arrow[d, "h^*"'] & map(Y,Z) \arrow[d, "f^*"] \\ map(X,Z) \arrow[r, "j^*"] & map(A,Z) \end{tikzcd}$$

is a pull back diagram. (This was the key step in proving that for a map $f : X \to Y$, $map(Z(f), Z) = P(f^*)$.)

**Zad. 8.** Present the map $X \coprod X \to X$ sending each summand identically onto $X$ as the composition of a cofibration and homotopy equivalence.
Present the diagonal map $\Delta : Y \to Y \times Y$ as the composition of a homotopy equivalence and a fibration.

**Zad. 9.** Present the map $X \to \{*\}$ as the composition of a cofibration and homotopy equivalence. Present the map $\{*\} \to X$ as the composition of a homotopy equivalence and a fibration.