Exponential attractors for planar shear flows with subdifferential boundary conditions in lubrication theory and friction contact problems

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Warszawa, May 10, 2012
We are interested in:

- Time asymptotics \((t \to \infty)\) of solutions of the NS shear flows, with subdifferential boundary conditions.
- Existence of global attractors,
- Dimension of global attractors
- Existence of exponential attractors
- Dependence of attractors’ dimension on the type of boundary conditions
- Turbulence studies
Examples. On the bottom part of the boundary:

\(u \cdot n = 0, \quad \sigma_T = \mu |u_T|^{p-1} u_T; \quad |\sigma_T| \leq k, k |u_T| + \sigma_T \cdot u_T = 0\)
**Classical formulation** of a class of problems:

\[ \frac{Du}{Dt} = \text{Div} \sigma + f \text{ in } \Omega, \quad \text{conservation law} \]

\( \sigma = \sigma(\epsilon) \text{ in } \Omega, \quad \text{constitutive law} \)

On the top part \( \Gamma_1 \) of the boundary \( u = 0 \)

On the lateral part \( \Gamma_L \): \( u \) is \( L \)-periodic in \( x_1 \)

On the **bottom** part \( \Gamma_0 \): \( u.n = 0 \) and

\[ \varphi(v) - \varphi(u) \geq -\sigma n(v - u) \text{ for all } v \in V \]

We have, for \( u, v \in V \), and the Navier-Stokes stress tensor \( \sigma \),
\[
- \int_{\Omega} \text{Div}(\sigma)(v - u)dx = \int_{\Omega} \epsilon(u) : \epsilon(v - u)dx - \int_{\Gamma_0} \sigma n \cdot (v - u)dS
\]
but
\[
- \int_{\Gamma_0} \sigma n \cdot (v - u)dS \leq \int_{\Gamma_0} (\varphi(v) - \varphi(u))dS
\]
and we obtain the desired variational inequality with a boundary functional
\[
j(v) = \int_{\Gamma_0} \varphi(v)dS,
\]
where, e.g., (1) \( \varphi(v) = \mu |v|^{p+1} \), (2) \( \varphi(v) = k |v| \).
On the bottom $\Gamma_0$: the normal component of the velocity, 
$u \cdot n = 0$.
The tangential component $u_\eta$ is unknown and satisfies the Tresca friction law with a constant and positive maximal friction $k$,

\[
|\sigma_\tau(u,p)| < k \Rightarrow u_\tau = U_0 e_1
\]

\[
|\sigma_\tau(u,p)| = k \Rightarrow \exists \lambda \geq 0 \text{ such that } u_\tau = U_0 e_1 - \lambda \sigma_\tau(u,p)
\]

where $\sigma_\tau$ is the tangential component of the stress tensor on $\Gamma_0$ and $U_0 e_1 = (U_0, 0)$, is the velocity of the lower surface, producing the driving force of the flow.
The variational formulation of the homogenized problem:

Problem

Given $v_0 \in H$, find $v : (0, \infty) \to H$ such that:

(i) for all $T > 0$, $v \in C([0, T]; H) \cap L^2(0, T; V)$, $v_t \in L^2(0, T; V')$

(ii) for all $\Theta$ in $V$, all $T > 0$, and for almost all $t$ in the interval $[0, T]$, the following variational inequality holds

$$< v_t(t), \Theta - v(t) > + \nu a(v(t), \Theta - v(t)) + b(v(t), v(t), \Theta - v(t))$$

$$+ j(\Theta) - j(v(t)) \geq (\mathcal{L}(v(t)), \Theta - v(t))$$

(iii) the initial condition $v(x, 0) = v_0(x) = u_0(x) - U_0(x)e_1$.

where, for example (2),

$$j(v(t)) = \int_{\Gamma_0} k|v(x_1, 0, t)|dx_1. \quad (\Gamma_0 \quad \text{bottom})$$
We recall the stress tensor of Bingham fluid,

\[
\sigma(v, p) = -pl + 2\nu D(v) + g \frac{D(v)}{|D(v)|} \quad \text{if} \quad |D(v)| \neq 0
\]

\[
|\sigma| \leq g \quad \text{if} \quad |D(v)| = 0
\]

where \( g \geq 0 \) represents the yield limit.

Then we consider the inequality (we take \( U_0 = 0 \), driving force \( f \))

\[
< v_t(t), \Theta - v(t) > + \nu a(v(t), \Theta - v(t)) + b(v(t), v(t), \Theta - v(t)) + J(\Theta) - J(v(t)) + j(\Theta) - j(v(t)) \geq (f, \Theta - v(t))
\]

where

\[
J(v) = g \int_{\Omega} |D(v)| \, dx \quad \text{and} \quad j(v) = \int_{\Gamma_0} k|v(x_1, 0)| \, dx_1.
\]
Theorem on the existence of solutions

**Theorem**

*For any initial velocity* $u_0$ *and velocity of the bottom boundary* $U_0$ *there exists a unique global in time solution of the considered initial-boundary value problem for the 2-dimensional Navier-Stokes.*

\[
< v_t(t), \Theta - v(t) > + \nu a(v(t), \Theta - v(t)) + b(v(t), v(t), \Theta - v(t)) \\
+ j(\Theta) - j(v(t)) \geq (\mathcal{L}(v(t)), \Theta - v(t))
\]
Basic notion: dynamical system

Let us consider a dissipative, infinite-dimensional dynamical system:

\[
\frac{du}{dt} = F(u) \quad (1)
\]

\[
u(0) = u_0 \in H \quad (H = \text{the phase space}) \quad (2)
\]

2D Navier-Stokes is a dissipative dynamical system.

We are interested in large time behaviour of solutions.

Solution: \( u(t) = S(t)u_0, \ t \geq 0, \) where \( \{S(t)\}_{t \geq 0} \) is a semigroup, \( S(t) : H \to H. \)

Properties of the semigroup \( \{S(t)\}_{t \geq 0} \) in \( H \) give information about behaviour of trajectories \( t \to u(t) \in H \) for large times \( t. \)
**Theorem on the existence of a global attractor**

**Theorem**

There exists a global attractor $\mathcal{A}$ for the associated semigroup $S(\cdot)v_0 = v(t)$ in $H$ for the considered problem.

\[ < v_t(t), \Theta - v(t) > + \nu a(v(t), \Theta - v(t)) + b(v(t), v(t), \Theta - v(t)) \]

\[ + \ j(\Theta) - j(v(t)) \geq (L(v(t)), \Theta - v(t)) \]

(Methods based on the notion of the Kuratowski measure of the noncompactness, norm-to-weak continuity,..., or the method of l-trajectories (Malek-Prazak, 2002))
Dimension of the attractor

**Theorem**

*The fractal dimension of the global attractor $\mathcal{A}$ is finite.*

Problems with the **smoothing property**:

$$\|S(t)x_1 - S(t)x_2\|_V \leq c\|x_1 - x_2\|_H$$

in the original phase space $H$, but

$L(t) : H_l \rightarrow W_l$ has this property

(where: $H_l = L^2(0, T; H)$, $W_l = \{u \in L^2(0, T; V), u' \in L^1(0, T, V')\}$)

**Thm** $\mathcal{X}$, $\mathcal{Y}$ -normed, $\mathcal{Y} \subset \subset \mathcal{X}$, $\mathcal{C} \subset \mathcal{X}$ bounded, and $\mathcal{L} : \mathcal{X} \rightarrow \mathcal{Y}$ Lipschitz, $\mathcal{L}(\mathcal{C}) \subset \mathcal{C}$. Then $d_f^{\mathcal{X}'}(\mathcal{C}) < \infty$. 
An exponential attractor is a subset $\mathcal{M}$ in the phase space $H$ such that:

- $\mathcal{M}$ is compact in $H$.
- $\mathcal{M}$ positively is invariant: $S(t)\mathcal{M} \subset \mathcal{M}$ for $t \geq 0$.
- $\mathcal{M}$ attracts bounded sets in $H$: $\text{dist}(S(t)B, \mathcal{M}) \to 0$ as $t \to \infty$ at an exponential rate.
- $\mathcal{M}$ has a finite fractal dimension.

The global attractor $\mathcal{A}$ is contained in $\mathcal{M}$.

$\mathcal{M}$ is not uniquely defined - there are several constructions.

$\mathcal{M}$ has good stability properties with respect to perturbations of the system.
Existence of an exponential attractor

**Theorem**

**There exists an exponential attractor** $\mathcal{M}$

*for the associated semigroup* $S(\cdot)\nu_0 = \nu(t)$ in $H$ *for the considered problem.*

Main problem: to prove $\|L(t_1)\chi - L(t_2)\chi\|_{H_l} \leq c|t_1 - t_2|^{\beta}$, to construct an exponential attractor, first for the discrete semigroup

$L(nt_0)$ in $H_l = L^2(0, l; H)$.

At last, we need $e : H_l \supset \mathcal{H}_l \to H$, $e(\chi) = \chi(l)$ to be Hölder continuous.

(Hölder continuous map preserves finiteness of the fractal dimension).
Let the external force be equal to $\lambda f$, where $\lambda > 0$ and $f \in H$ and

$$
\lambda_* = \inf \{ g \int_{\Omega} |D(v)| dx + k \int_{\Gamma_0} |v(x_1, 0)| dx_1 : v \in V, \int_{\Omega} fv dx = 1 \}. 
$$

Then the following theorem holds.

**Theorem**

For $\lambda \in [0, \lambda_*]$ the global attractor is trivial, $A = \{0\}$, which means that it coincides with the unique stationary solution $v = 0$. Moreover, if $0 \leq \lambda < \lambda_*$ then for any bounded set $B$ in $H$ there exists a time $t(B)$ such that $|S(t)v_0| = 0$ for all $v_0 \in B$ and all $t > t(B)$. For $\lambda = \lambda_*$ we have, for all $t \geq 0$ and some $\gamma > 0$,

$$
|S(t)v_0| \leq |v_0| e^{-\gamma t}. \tag{3}
$$

In particular, for $\lambda \in [0, \lambda_*]$ the global attractor itself is an exponential attractor.
The proof uses the fact that $g > 0$. Passing with $g$ to zero means passing from the Bingham model to the Navier-Stokes model, and for the latter we cannot obtain such a theorem. From the physical point of view this is connected with plastic properties of the Bingham fluid.
We consider the stability of the global attractor for the Navier-Stokes model with respect to a perturbation introduced by the functional $J$. Let $A_g$ be the global attractor for the Bingham fluid with the yield limit $g > 0$, and $A_0$ be the global attractor for the Navier-Stokes model. We prove that the attractors $A_g$ for $0 \leq g < g_0$ are uniformly bounded in $H$, and also that

$$\text{dist}(A_g, A_0) \to 0 \quad \text{as} \quad g \to 0,$$

which is the upper semicontinuity property.
Open and interesting problems

- To consider other evolutionary variational inequalities for dynamic and quasistatic contact problems.


THANK YOU