Istnienie klasycznych rozwiązań dla kwantowego równania Boltzmanna dla fermionów

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$$\partial_t f(x,k,t) + k\nabla f(x,k,t) = Q(f), \tag{1}$$

and the collision operator Q as derived in $[H]$ has a form

$$Q(f) = \pi \int d_3l d_3m d_3n| < lm|V|nk > |^2 \delta^3(l+m-n-k)\delta(\epsilon_l, +\epsilon_m - \epsilon_n - \epsilon_k)$$

$$\{f(x,l,t)f(x,m,t)[1-f(x,n,t)][1-f(x,k,t)] -$$

$$f(x,k,t)f(x,n,t)[1-f(x,l,t)][1-f(x,m,t)]\} . \tag{2}$$

$$Q(f) = 4\pi \int d_3k_1 d_3k' d_3k'_1 | < kk_1(\tau^+_{12}(\epsilon_{12};t)2^{-1}(1+\pi_{12})|k'k'_1 > |^2$$
$$\delta(\epsilon_{12}-\epsilon'_{12})\left\{f'f'_1(1-f)(1-f_1)-(1-f')(1-f'_1)ff_1\right\}, \tag{1}$$

where $f = (x,k,t), f_1 = (x,k_1,t), f' = (x,k',t), f'_1 = (x,k'_1,t)$ and k, k_1. describe momentum of the particles before and k', k'_1 after scattering $\epsilon_{12} = m^{-1}(k^2+k_1^2), \epsilon'_{12} = m^{-1}(k'^2+k'^2_1)$. The exchange modified t-matrix is defined as

$$\tau^+_{12}(E,t) = \lim_{\epsilon\to 0^+} V_{12}\left[\epsilon + i[H(12;t)-E]\right]^{-1}\left[\epsilon + i[H_0(12)-E]\right] \tag{2}$$

where $H_0(12)$ is two particle kinetic energy and $H(12;t) = H_0(12) + S_{12}V_{12}$ and $S_{12}(t) = 1-\rho_1(t)-\rho_2(t)$. $\rho_i(t)$ is a one particle density operator and $V_{ij} = V(|x_i - x_j|)$ describes interactions.

$$\partial_t f(x,\xi,t) + \xi\nabla f(x,\xi,t) = \int d_3\xi_1 d_2 e B(\xi,\xi_1,\Theta)\left\{f(x,\xi',t)f(x,\xi_1',t)\right.$$

$$[1-f(x,\xi,t)][1-f(x,\xi_1,t)] - f(x,\xi,t)f(x,\xi_1,t)[1-f(x,\xi',t)] \qquad (1)$$

$$\left.[1-f(x,\xi_1',t)]\right\},$$

where ξ,ξ_1 are momenta before and ξ',ξ_1' after collision, e is a unit vector in the direction of center of particles during collision, $Theta$ is a scattering angle and the collision kernel B has a following form

$$B(\xi,\xi_1,\Theta) = 2^{-1}\sigma(|\xi-\xi_1|,\Theta)|\xi-\xi_1|, \qquad (2)$$

in nonrelativistic case and

$$B(\xi,\xi_1,\Theta) = \xi_0^{-1}\xi_{10}^{-1}2^{-1}gs^{1/2}\sigma(g,\Theta), \qquad (3)$$

i) $B(\eta,\theta) \in L^1(R^3 \times S^2)$,
ii) $f_0(x,\xi) \geq 0$ a.e. in $R^3 \times R^3$; $f_0 \in L^1(R^3 \times R^3)$; $||f_0||_\infty \leq 1$.

Then the following holds

Theorem:
Exists function $f \in C([0,\infty[; L^1(R^3 \times R^3)$ such that $f \geq 0$ a.e. in $R_+ \times R^3 \times R^3$, $f_{|t=0} = f_0$. f is a mild or equivalently distributional solution to Eq.(1, previous slide) and satisfies following estimates

$$||f(t)||_1 = ||f_0||_1, \tag{1}$$

$$||f(t)||_\infty \leq 1. \tag{2}$$

$$\partial_t f^{N+1} + \xi \nabla f^{N+1} = -(h_1^N + h_2^N) f^{N+1} + h_2^N, \tag{1}$$

$$f^{N+1}(t=0) = f_0. \tag{2}$$

The solution to the Eq.(1) can be written as

$$f^{N+1}(x,\xi,t) = \exp\left[-\int_0^t d\tau [h_1^N(x-(t-\tau)\xi,\xi,\tau) + h_2^N(x-(t-\tau)\xi,\xi,\tau)]\right]$$

$$f_0(x-t\xi,\xi) + \int_0^t ds \left\{\exp\left[-\int_s^t d\tau [h_1^N(x-(t-\tau)\xi,\xi,\tau) + h_2^N(x-(t-\tau)\xi,\xi,\tau)]\right]\right.$$

$$\left. h_2^N(x-(t-s)\xi,\xi,s)\right\} \tag{3}$$

For f^{N+1} we have following a priori bounds providing that $h_i^N \in L^\infty$ and $h_i^N \geq 0$ a.e.

$$f^{N+1} \leq \exp\left[-\int_0^t d\tau [h_1^N(x-(t-\tau)\xi,\xi,\tau) + h_2^N(x-(t-\tau)\xi,\xi,\tau)]\right] f_0(x-\xi t,\xi)$$

$$+1 - \exp\left[-\int_0^t d\tau [h_1^N(x-(t-\tau)\xi,\xi,\tau) + h_2^N(x-(t-\tau)\xi,\xi,\tau)]\right] \tag{4}$$

we see that if $f_0 \leq 1$ then

$$f^{N+1} \leq \text{ a.e. in } R_+ \times R^3 \times R^3, \tag{1}$$

$$||h_{1,2}^{N+1}||_\infty \leq ||B||_1, \tag{2}$$

$$||h_{1,2}^{N+1}||_1 \leq ||B||_1 ||f(t)||_1, \tag{3}$$

and immediately obtain following global bound on $||f^N(t)||_1$

$$||f^N(t)||_1 \leq ||f_0||_1 \exp[||B||_1 t]. \tag{4}$$

We obtain existence of solution as a direct application of Kolmogorow-Riesz theorem and Arzelá-Ascoli theorem.

Now the mild solution to the QBE has a form

$$f(x,\xi,t) = f_0(x - t\xi, \xi) + \int\limits_0^t ds Q[f(x - t - s)\xi, \xi, s], \tag{1}$$

Theorem: Assume that $f_0 \in L^1(R^3 \times R^3) \cap L^\infty(R^3 \times R^3)$ with $||f_0||_\infty \leq 1$ such that

$$||\xi^2 f_0||_1 < \infty, \tag{1}$$

$$||x^2 f_0||_1 < \infty, \tag{2}$$

then the solution $f(t)$ of the QBE fulfils following estimates

$$||\xi^2 f(t)||_1 = ||\xi^2 f_0||_1, \tag{3}$$

$$||(x - \xi t)^2 f(t)||_1 = ||x^2 f_0||_1 \tag{4}$$

Theorem: The mild solution to QBE is unique.

Theorem: Assume $f_0 \in L^1 \cap L^\infty(R^3 \times R^3)$ with $||f_0||_\infty \leq 1$ and such that $||\nabla f||_1 < \infty$. Let $f \in C([0,\infty[; L^1(R^3 \times R^3))$ be solution of the QBE such that $f(0) = f_0$ then $\nabla f \in C([0,\infty[; L^1(R^3 \times R^3))$ for any $t < \infty$.

Theorem: Assume $f_0 \in L^1 \cap L^\infty(R^3 \times R^3)$ with $||f_0||_\infty \leq 1$ and such that $||f_0 \ln f_0 + (1 - f_0)\ln(1 - f_0)||_1 < \infty$.
Let $f \in C([0,\infty[; L^1(R^3 \times R^3))$ be solution of the QBE such that $f(0) = f_0$ then $f \ln f + (1 - f)\ln(1 - f) \in C^1([0,\infty[; L^1(R^3 \times R^3))$ and
$\frac{d}{dt}||f \ln f + (1 - f)\ln(1 - f)||_1 \leq 0$.

Let as assume

$$B \in L^1_{loc}(R^3 \times S^2), \tag{1}$$

$\exists \epsilon > 0, \eta_0 > 0, C > 0$ such that for $|\eta| > \eta_0$

$$\int d\Theta B(\eta, \Theta) < C\eta^{2-\epsilon}, \tag{2}$$

and

$$0 \leq f_0 \leq 1 \text{ a.e. in } R^3 \times R^3 \tag{3}$$

$$f_0 \in L^1(R^3 \times R^3), \tag{4}$$

$$\xi f_0 \in L^1(R^3 \times R^3), \tag{5}$$

$$f_0 \ln f_0 + (1 - f_0)\ln(1 - f_0) \in L^1(R^3 \times R^3). \tag{6}$$

Then we have the following

Theorem: Assume (1-2) and (3-6) then exists $f \in C([0, \infty[; L^1(R^3 \times R^3))$ such that f is a mild or equivalently distributional solution to QBE with collision kernel B. Moreover the following assertions hold
i) $Q^{\pm}[f] \in C([0, \infty[; L^1(R^3 \times R^3))$, for any $r > 0$,
ii) $||f(t)||_1 = ||f_0||_1$,
iii) $||\xi^2 f(t)||_1 = ||\xi^2 f_0||_1$,
iv) $||f(t)\ln f(t) + (1 - f(t))\ln(1 - f(t))||_1 = ||f_0 \ln f_0 + (1 - f_0)\ln(1 - f_0)||_1$.

Let us assume

$$\int d_3(p-p_1)d(\cos\theta)B(p,p_1,\theta)<\infty, \tag{1}$$

$\exists\epsilon>0,\eta_0>0,C>0$ such that for $|p-p_1|>\eta_0$

$$\int d(\cos\theta)B(p,p_1,\theta)<C(p-p_1)^{1-\epsilon}, \tag{2}$$

and

$$0\leq f_0\leq 1 \text{ a.e. in } R^3\times R^3 \tag{3}$$

$$f_0\in L^1(R^3\times R^3), \tag{4}$$

$$p_0f_0\in L^1(R^3\times R^3), \tag{5}$$

$$f_0\ln f_0+(1-f_0)\ln(1-f_0)\in L^1(R^3\times R^3). \tag{6}$$

Theorem: Consider RQBE with collision kernel $B(p,p_1,\theta)$ which fulfills (1,2) and let f_0 fulfils (3-6) then exists $f\in C([0,\infty[;L^1(R^3\times R^3))$ such that f is a mild or equivalently distributional solution to the RQBE and the following assertions hold
i) $f_{|t=0}=f_0$
ii) $0\leq f_0\leq 1$, a.e. in $R_+\times R^3\times R^3$,
iii) $||f(t)||_1=||f_0||_1$,
iv) $||p_0f(t)||_1||p_0f_0||_1$,

if in addition $\int d_3(p-p_1)d(\cos\theta)B(p,p_1,\theta)<\infty$ then the solution is also unique, and we can drop condition (5,6).

Assume that the collision kernel $B(z, \eta, \theta)$ satisfies

$$|\partial_z B(z, \eta, \theta)| \leq \hat{B}(\eta, \theta), \text{ for } |z| \leq 1, \tag{1}$$

where

$$\hat{B}(\eta, \theta) \in L^1(R^3 \times S^2). \tag{2}$$

Then we have following extension of the theorem: Exists function $f \in C([0, \infty[; L^1(R^3 \times R^3)$ such that $f \geq 0$ a.e. in $R_+ \times R^3 \times R^3$, $f_{|t=0} = f_0$. f is a mild or equivalently distributional solution to Eq.(1, previous slide) and satisfies following estimates

$$||f(t)||_1 = ||f_0||_1, \tag{3}$$

$$||f(t)||_\infty \leq 1. \tag{4}$$

Extension:

Theorem: Let B fulfills conditions (1) and (2) and let f_0 fulfills (1,2 on previous slide). Then exists $f \in C([0, \infty[; L^1(R^3 \times R^3))$ such the f is a unique mild or equivalently distributional solution to the QBE with collision kernel B.