Collective dynamics of interacting particles

Ph.D. Thesis Extended Abstract

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Flocking, swarming, aggregation - there is a multitude of actual real-life phenomena that from the mathematical point of view can be interpreted as one of these concepts. The mathematical description of collective dynamics of self-propelled agents with nonlocal interaction originates from one of the basic equations of kinetic theory – Vlasov’s equation from 1938. Recently it was noted that such models provide a way to describe a wide range of phenomena that involve interacting agents with a tendency to aggregate their certain qualities. This approach proved to be useful and the language of aggregation now appears not only in the models of groups of animals but also in the description of seemingly unrelated phenomena such as the emergence of common languages in primitive societies, distribution of goods or reaching a consensus among individuals [3,31,33,42]. The literature on such aggregation models is very rich thus we will only mention a few examples on some of the more popular branches of the research. Those branches include the analysis of time asymptotics (see e.g. [27]) and pattern formation (see e.g. [26,41]) or analysis of the models with additional forces that simulate various natural factors (see e.g. [11,20] - deterministic forces or [15] - stochastic forces). The other variations of the model include forcing particles to avoid collisions (see e.g. [13]) or to aggregate under the leadership of certain individuals (see e.g. [14]). A good example of a paper in which a well rounded analysis of a model that includes effects of attraction, repulsion and alignment is [7]. The general form of equations associated with aggregation models reads as follows:

$$\partial_t f + v \cdot \nabla f + \text{div}_v[(k \ast f)f] = 0,$$

where $f = f(x, v, t)$ is usually interpreted as the density/distribution of particles at the time $t$ with position $x$ and velocity $v$. Function $k$ is the kernel of the potential generating the motion. It is responsible for the non-local interaction between particles and depending on it the particles may exhibit various tendencies like to flock, aggregate or to disperse. The common properties required from kernel $k$ in most models include Lipschitz continuity and boundedness and it is the case due to the fact that many standard methods work well with such assumptions. For instance if $k$ is Lipschitz continuous and bounded then the particle system associated with (1) is well–posed, the characteristic method can be performed for (1) and one can usually pass from the particle system to the kinetic equation by mean-field limit. The main goal of this thesis is to consider $k$ that is singular and refine the mean-field limit approach to be applicable in such scenario. We study this problem in a particular case of the Cucker-Smale (C–S) flocking model.

**The Cucker–Smale flocking model.** In [16] from 2007, Cucker and Smale introduced a
model for the flocking of birds associated with the following system of ODEs:

\[
\begin{align*}
\frac{dx_i}{dt} &= v_i, \\
\frac{dv_i}{dt} &= \sum_{j=1}^{N} m_j (v_j - v_i) \psi(|x_j - x_i|),
\end{align*}
\]

where \( N \) is the number of the particles while \( x_i(t) \), \( v_i(t) \) and \( m_i \) denote the position and velocity of \( i \)th particle at the time \( t \) and its mass, respectively. Function \( \psi : [0, \infty) \to [0, \infty) \) usually referred to as the communication weight is nonnegative and nonincreasing and can be vaguely interpreted as the perception of the particles. The communication weight plays a crucial role in our investigations and we will focus on it more in a while. We refer to system (2) as the C–S particle system or the discrete C–S model (sometimes we omit 'C–S').

As \( N \to \infty \) the particle system is replaced by the following Vlasov-type equation:

\[
\begin{align*}
\frac{\partial f}{\partial t} + v \cdot \nabla f + \text{div}_v [F(f)f] &= 0, \quad x \in \mathbb{R}^d, \quad v \in \mathbb{R}^d, \\
F(f)(x, v, t) &= \int_{\mathbb{R}^{2d}} \psi(|y - x|)(w - v)f(y, w, t) dw dy,
\end{align*}
\]

which can be written as (1) with \( k(x, v) = v\psi(|x|) \). As mentioned before we are considering (3) with a singular kernel

\[
\psi(s) = \begin{cases} 
    s^{-\alpha} & \text{for } s > 0, \\
    \infty & \text{for } s = 0,
\end{cases} \quad \alpha > 0.
\]

We refer to equation (3) as the kinetic C–S equation, the Vlasov-type C–S equation or the continuous C–S model (sometimes we omit 'C–S').

The story of C–S model should probably begin with [43] by Vicsek et al., where a model of flocking with nonlocal interactions was introduced and it is widely recognized to be up to some degree an inspiration for [16]. Since 2007 the C–S model with a regular communication weight of the form

\[
\psi_{cs}(s) = \frac{K}{(1 + s^2)^{\beta}}, \quad \beta \geq 0, \quad K > 0
\]

was extensively studied in the directions similar to those of more general aggregation models (i.e. collision avoiding, flocking under leadership, asymptotics and pattern formation as well as additional deterministic or stochastic forces - see [2, 10, 25, 28, 35, 40]). Particularly interesting from our point of view is the case of passage from the particle system (2) to the kinetic equation (3), which in case of the regular communication weight was done for example in [29] or [30]. For a more general overview of the passage from microscopic to mesoscopic and macroscopic descriptions in aggregation models of the form (1) we refer to [8, 17, 18]. The C–S model with singular weight was introduced in 2009 in [29] and the main contributions to its analysis are papers [1, 9, 36, 37].
Main result. The main goal of the dissertation is to prove that for any initial Radon measure \( f_0 \) and \( T > 0 \) the Vlasov-type C–S equation (3) with singular weight given by (4) admits solutions in the interval \([0, T]\), provided that the range of singularity of \( \psi \) is less than 1 (i.e. \( \alpha \in (0, 1) \)).

The general strategy is standard and can be summarized as follows. It is natural to expect that the analysis (be it qualitative or quantitative) of the particle system (2) is more approachable than in the continuous case. We define the solutions to (3) by approximation with the solutions of (2) with the number of particles \( N \) going to infinity. We use the mean-field limit approach. Given a Radon measure \( f_0 = f_0(x, v) \), where \( x \in \mathbb{R}^d \) and \( v \in \mathbb{R}^d \) as an initial datum, we divide it’s support into congruent cubes \( Q_{i,\epsilon} \subset \mathbb{R}^d \) of diameter \( \epsilon > 0 \) (the centres of the cubes, denoted by \((x_{i,\epsilon}, v_{i,\epsilon})\), form a lattice-shaped \( \epsilon \)-net on the support of \( f_0 \)). In the centre of each of the cubes we place a Dirac’s delta of a mass \( m_{i,\epsilon} \) equal to the total mass of \( f_0 \) restricted to the cube i.e.

\[
m_{i,\epsilon} := \int_{Q_{i,\epsilon}} f_0(x, v) dx dv.
\]

This way, denoting the number of cubes by \( N_{\epsilon} \), we obtain

\[
f_{0,\epsilon} := \sum_{i=1}^{N_{\epsilon}} m_{i,\epsilon} \delta_{x_{i,\epsilon}} \otimes \delta_{v_{i,\epsilon}},
\]

which we prove that converges\(^1\) to \( f_0 \) as \( \epsilon \to 0 \). However, alternatively we may look at the Dirac’s deltas \( m_{i,\epsilon} \delta_{x_{i,\epsilon}} \otimes \delta_{v_{i,\epsilon}} \) as a description of starting points of the particles in the system (2), where \( x_{i,\epsilon}, v_{i,\epsilon} \) and \( m_{i,\epsilon} \) denote the initial position and velocity and the mass of \( i \)th particle, respectively. Then the solution of the particle system (denote it by \((x_\epsilon, v_\epsilon)\)) can be again interpreted as measure valued function \( f_\epsilon \) from the time interval \([0, T]\). Then we converge with \( \epsilon \to 0 \) and hopefully extract a subsequence that converges to some measure valued function \( f \), which serves as a candidate for the solution to (3). This general strategy is utilised for example in \([29]\), where the authors prove well-posedness for the kinetic equation (3) with a regular communication weight. However they strongly relay on the Lipschitz continuity and boundedness of \( \psi \), which allows them to obtain well-posedness for the particle system, which makes the convergence with the approximate solutions \( f_\epsilon \) straightforward. On the other hand in case of singular communication weight there is little hope for the well-posedness for the particle system, which in turn makes extraction of the convergent subsequence much more difficult.

We apply this strategy in the following way. At the beginning we focus on the case of existence for the particle system (2) with the range of singularity \( \alpha \in (0, 1) \). We prove that for any initial data in the form of finite number of particles there exists piecewise–weak\(^2\)

\(^1\)We define the appropriate topology later.
\(^2\)The precise definition of piecewise–weak solutions is quite involved thus we skip it here and present it in the dissertation.
solution with various useful structural properties. We also provide an example of a solution with trajectories that stick together in a finite time (such phenomenon cannot occur in the case of regular communication weight). These results can be summarised as follows.

**Theorem 1.** Let \( \alpha \in (0, 1) \). For all \( T > 0 \) there exists a \((C^1([0, T]))^\mathbb{N}\) piecewise-weak solution of (2) with arbitrary initial data.

**Proposition 1.** The C–S particle system (2) with singular communication weight (4) with \( \alpha \in (0, 1) \) allows sticking of the trajectories of the particles.

Next, we strengthen Theorem 1 proving that by restricting the range of admissible \( \alpha \) to \((0, \frac{1}{2})\) we obtain existence and uniqueness of strong solutions to the particle system (2). This is the closest result to the (still open problem of) well-posedness for the C–S particle system with a singular communication weight. We present it in the form of the following theorem.

**Theorem 2.** Let \( \alpha \in (0, \frac{1}{2}) \) be given. Then for all \( T > 0 \) and arbitrary initial data there exists a unique \( x \in W^{2,1}([0, T]) \subset C^1([0, T]) \) that solves (2) with communication weight given by (4) weakly in \( W^{2,1}([0, T]) \).

Additionally we prove that the piecewise-weak solutions from Chapter 3 are unique for \( \alpha \in (0, 1) \).

**Theorem 3.** Let \( \alpha \in (0, 1) \) be given. Then the piecewise-weak solution of (2), which existence is ensured by Theorem 1 is unique.

We apply the above results (in particular Theorem 2) to obtain existence for the kinetic equation (3) with a singular weight with \( \alpha \in (0, \frac{1}{2}) \). We adopt the mean-field limit method that we sketched at the beginning of this section. The restriction of admissible \( \alpha \) to the interval \((0, \frac{1}{2})\) comes directly from Theorem 2 and should be understood in the following way: in order to obtain existence for the kinetic equation by mean-field limit one needs sufficient regularity of solutions to the particle system that grant compactness of the sequences of approximate solutions. In other words, our technique works if only the solutions to the particle system are regular enough.

**Theorem 4.** Let \( 0 < \alpha < \frac{1}{2} \). For any nonnegative compactly supported Radon measure \( f_0 \) as the initial data and any \( T > 0 \), the Cucker-Smale’s kinetic equation (3) admits at least one measure solution \( f \). Moreover if \( f_0 \) is atomic\(^4\), then \( f \) is atomic and it is unique.

**Flocking particles embedded in a fluid.** One of other directions of research is the analysis of the motion of agents (described by a kinetic model of the type (1)) in their natural habitat. Hence, parallely to the analysis of the kinetic models themselves, research in coupling models 3

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3 Precise definition included in the dissertation
4 By 'atomic' we understand that \( f_0 \) is a sum of Dirac’s deltas.
of kinetic theory with models of hydrodynamics was performed (see [4–6, 12, 22–24]). From the point of view of this thesis the most important examples of such research is the paper [6] in which the coupling of Navier-Stokes system (N–S) with Vlasov equation is considered and the paper [4] in which the approach of [6] is applied to N–S coupled with C–S (since C–S equation is actually a Vlasov-type equation). The secondary goal of this dissertation is to modify the approach used in [6] and [4] and couple C–S model with models of non-Newtonian fluids, which up to this point was not done.

Our goal is to consider particles embedded in an incompressible, viscous, non-Newtonian shear thickening fluid, i.e. we aim to couple (2) with the system

\[
\begin{align*}
\partial_t u + (u \cdot \nabla)u + \nabla \pi - \text{div} (\tau) &= f_{\text{ext}}, \\
\text{div} u &= 0,
\end{align*}
\] (6)

which describes a motion of such fluid in \( \mathbb{R}^d \). The function

\[ u = u(t, x) = (u_1(t, x), u_2(t, x), \ldots, u_d(t, x)) \]

represents velocity of the fluid at the position \( x \) and time \( t \). Equation (6)\(_2\) expresses the conservation of mass, while (6)\(_1\) expresses the conservation of momentum. The term \( \tau \) in (6)\(_1\) denotes a symmetric stress tensor that depends on \( Du \) – the symmetric part of the gradient of \( u \) i.e. \( \tau = \tau(Du) \), where \( Du = \frac{1}{2}[\nabla u + (\nabla u)^T] \). In our considerations we assume that \( \tau \) is derived from some scalar potential \( \vartheta \) and through requiring some specific properties of \( \vartheta \) we actually impose various other assumptions on \( \tau \) including \( p \)-growth (for some given \( p > 1 \)) or coercivity. Lastly function \( f_{\text{ext}} \) represents an external force.

The coupling of (2) with (6) is done via the drag force

\[ F_d(t, x, v) := u(t, x) - v, \]

that influence the motion of the particles and the fluid. This way of coupling and such drag force is adopted from [6] and [4] and originally it was used for the modelling of thin spray and fluid (see also [5, 6, 22–24]). Explicitly, the coupled system reads as follows:

\[
\begin{align*}
\partial_t f + v \nabla f + \text{div}_x [(F(f) + F_d)f] &= 0, \quad x \in \mathbb{T}^d, \quad v \in \mathbb{R}^d, \\
\partial_t u + (u \cdot \nabla)u + \nabla \pi - \text{div}(\tau(Du)) &= -d \int_{\mathbb{R}^d} F_d f dv, \quad x \in \mathbb{T}^d, \\
\text{div} u &= 0.
\end{align*}
\] (7)

Let us briefly discuss the difference between coupling of the C–S model with Newtonian and non-Newtonian fluids. In [6] and [4], the authors obtained weak existence for their coupled systems and on top of that in [4], the authors obtained asymptotic flocking. In particular there was little hope to obtain regularity or uniqueness for coupled N-S-Vlasov or N-S-C-S systems without previously obtaining it for N-S system. However in case of coupling with a non-Newtonian fluid, existence, regularity and possibly uniqueness depend on the value of the exponent \( p \) and regularity of the external function \( f_{\text{ext}} \). For uncoupled non-Newtonian
system (6) weak existence is known for $p > \frac{2d}{d+2}$ and $f_{\text{ext}} \in (W^{1,p})'$. It is obtained by Lipschitz truncation method (see [19, 21]). On the other hand if $p \geq \frac{3d+2}{d+2}$ and $f_{\text{ext}} \in L^2(0,T;L^2(\mathbb{T}^d))$, not only do we have existence of strong solutions but also uniqueness (see [39]). Therefore, on top of the interesting asymptotics similar to those from the paper [4], we may expect the possibility of better regularity and of uniqueness for the coupled system. However it depends on $p$ and the structure of the external force, which in our case equals

$$f_{\text{ext}} = -d \int_{\mathbb{R}^d} (u-v)f dv.$$  

Moreover for $p \in (\frac{2d}{d+2}, \frac{3d+2}{d+2})$ in case of the coupled system it appears that a combination of our approach with the Lipschitz truncation method should make obtaining existence of weak solutions possible but it is outside of our scope.

The general strategy of the proof is taken from [6] and executed with the help of results from papers [4, 32, 39]. First we regularise the system, then to obtain existence for the regularised system we introduce a inductive scheme solving C–S and fluid parts of (7) alternating with every step. Thus we solve (7)\textsuperscript{1} and put it’s solution into (7)\textsuperscript{2} as a given function then solve again obtaining a solution which we again put into (7)\textsuperscript{1}, solve and so on. Then the convergence of the approximations is obtained through a careful technical estimation and analysis.

Applying this strategy results in the following theorem.

**Theorem 5.** Let $d = 3$, $p \geq \frac{11}{5}$ and $T > 0$ and suppose that the initial data $(f_0, u_0)$ satisfy

1. $0 \leq f_0 \in (L^1 \cap L^\infty)(\mathbb{T}^3 \times \mathbb{R}^3)$, $\text{supp}f_0(x, \cdot) \subset B(R)$ for some $R > 0$ and a.a. $x \in \mathbb{T}^3$, where $B(R)$ is a ball centred at 0 with radius $R$,

2. $u_0 \in W^{1,2}(\mathbb{T}^3) \cap L^2_{\text{div,0}}(\mathbb{T}^3),$

3. $\nabla_r f_0 \in L^3(\mathbb{T}^3 \times \mathbb{R}^3).$

Then there exists a unique strong solution of (7) with regular communication weight.

In the above theorem $L^2_{\text{div,0}}$ represents the divergence–free subspace of $L^2(\mathbb{T}^3)$.

**Conclusions.** The following thesis presents my contribution in the development of the existence theory for the C–S model with a singular communication weight. It should be viewed as a step towards well-posedness for this system. Since 2014 we have managed to make a first successful attempt on proving:

- existence of piecewise-weak solutions to the C–S particle system for the range of singularities $\alpha \in (0, 1)$, published in [36];

- existence and uniqueness of strong solutions to the C–S particle system for the range of singularities $\alpha \in (0, \frac{1}{2})$, published in [37];
• existence and conditional uniqueness of solutions to the C–S kinetic equation for the range of singularities $\alpha \in (0, \frac{1}{2})$, included in the preprint [34];

• possibility of sticking of the trajectories of the particles, published also [36].

To our best knowledge these are the first results on existence for the C–S model with singular weight with $\alpha \in (0, 1)$ and some of the first steps in the direction of well-posedness for this model, which is important from the point of view of applications and numeric analysis. Due to the lack of existing theory we had to develop relatively new (often elementary) techniques or to significantly modify the existing ones. The analysis of the particle system was performed with the methods that originated from elementary techniques of the theory of systems of ODE’s. On the other hand the analysis of the kinetic equation was done by passing from the particle to kinetic description. Such passage was performed using much more sophisticated methods: originating from the stochastic analysis of complex many-body systems, mean-field limit method, was used to establish the kinetic equation as a limiting case of the particle system. The topology in which the limiting process was performed was generated by the Wasserstein W1 metrics (sometimes referred to as Kantorovich-Rubinstein metric or bounded–Lipschitz distance). In order to apply this method in the case of singular communication weight we had to significantly modify it.

Additionally we obtained

• existence and uniqueness of strong solutions to the C–S kinetic equation with a regular communication weight coupled with equations of non-Newtonian shear thickening fluids. This work is based on the modified methods from [6] and [4] coupled with results from [39] and [32] and is included in the preprint [38].

References


