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Warsaw University
Faculty of Mathematics, Informatics and Mechanics

Exam for 2nd cycle studies of MACHINE LEARNING

1st July 2021

Solving time: 150 minutes

In each of the 30 problems there are three variants: (a), (b), and (c). For each variant you should answer if it is true, writing **YES** *or* **NO** *in the box close to it. In case of error you should cross out the box and write the correct word on its left side.*

Example of a correctly solved problem

4. Every integer of form $10^n - 1$, where n is integer and positive,

$\boxed{\text{YES}}$ (a) is divisible by 9;
$\boxed{\text{NO}}$ (b) is prime;
$\boxed{\text{YES}}$ (c) is odd.

You can write only in the indicated places and only the words **YES** *and* **NO**. *Use pen.*

Scoring

You get "big" points (0 – 30) and "small" points (0 – 90):

- *one "big" point for each problem, in which you correctly solved all three variants;*
- *one "small" point for each correctly solved variant. So 3 "small" points in a single problem give one "big" point.*

The final result of the exam is the number

$$W = D + m/100$$

where D is the number of "big" points, and m is the number of "small" points, e.g. score 5.50 means that a candidate correctly solved 50 variants in the whole test, but gave correct answers to all three variants in a set for some five problems.

"Big" points are more important. "Small" points are just to increase resolution in case when many candidates get the same number of "big" points.

Good luck!

1. Each bounded subset of the rational numbers

- [ ] (a) has supremum, which is a rational number;
- [ ] (b) has supremum, which is a real number;
- [ ] (c) is contained in a union of open intervals I_j of lengths $|I_j| = d_j$, $j = 1, 2, \ldots$, satisfying inequality $\sum_{j=1}^{\infty} d_j < 1/2021!$.

2. Assume that $a_n > 0$ for $n \in \mathbb{N}$ and series $\sum_{n=1}^{\infty} a_n^2$ is convergent. It follows that

- [ ] (a) series $\sum_{n=1}^{\infty} \frac{a_n}{\sqrt{n}}$ is convergent;
- [ ] (b) series $\sum_{n=1}^{\infty} a_n^3$ is convergent;
- [ ] (c) $\lim_{n\to\infty} \sqrt{a_n} = 0$.

3. Given a real parameter $p > 0$ we define the function $f_p(x) = \frac{1-\cos x}{|x|^p}$ for $x \neq 0$. Let $g_p = \lim_{x\to 0} f_p(x)$. It follows that

- [ ] (a) $g_p \in \mathbb{R}$ for exactly one $p > 0$;
- [ ] (b) there exists an unbounded set $I \subset \mathbb{R}_+$ such that $g_p \in \mathbb{R}$ for $p \in I$;
- [ ] (c) if $g_{p^*} \in \mathbb{R}$ for some $p^* > 0$ then $g_p \in \mathbb{R}$ for $0 < p < p^*$.

4. Consider a function $f : U \to \mathbb{R}$ where $U \subset \mathbb{R}^m$ is an open set and $x_0 \in U$. It follows that

- [ ] (a) if all partial derivatives of f exist at x_0 then f is continuous at x_0;
- [ ] (b) if f is differentiable at x_0 then all partial derivatives of f are continous at x_0;
- [ ] (c) if all partial derivatives of f exist at x_0 then the directional derivative of f exists at x_0 in any direction $h \in \mathbb{R}^m$.

5. Let $\mathbb{G} = (G, \circ, e)$ be a group. Consider the group $\mathbb{G}^{\mathrm{op}} = (G, \cdot, e)$, where $x \cdot y = y \circ x$ for all $x, y \in G$. It follows that

- [ ] (a) the function $f\colon G \to G\colon x \mapsto x^{-1}$ is a homomorphism $\mathbb{G} \to \mathbb{G}^{\mathrm{op}}$;
- [ ] (b) groups $\mathbb{G}$ and $\mathbb{G}^{\mathrm{op}}$ are isomorphic;
- [ ] (c) $\mathbb{G} = \mathbb{G}^{\mathrm{op}}$ if and only if $\mathbb{G}$ is abelian.

6. Assume that some 3 rows of a matrix $A \in \mathbb{R}^{17,17}$ are linearly independent in $\mathbb{R}^{17}$. It follows that

- [ ] (a) the dimension of the image of matrix A is at least 3;
- [ ] (b) the dimension of the kernel of matrix A is at least 3;
- [ ] (c) some 14 columns of matrix A are linearly independent in $\mathbb{R}^{17}$.

7. Let X, Y be linear spaces. For linear maps $f\colon X \to Y$ and $g\colon Y \to X$ it holds that $g \circ f = 1_X$, where $1_X(x) = x$ for all $x \in X$. It follows that

- [ ] (a) the kernel of map f is the zero linear space;
- [ ] (b) the kernel of map g is the zero linear space;
- [ ] (c) map g is an epimorphism.

8. Which of the following subsets of the linear space $\mathbb{C}[x]$ of polynomials over the field $\mathbb{C}$ is a linear subspace?

- [ ] (a) $\{p \in \mathbb{C}[x] : \deg(p) = 2021\}$;
- [ ] (b) $\{p \in \mathbb{C}[x] : \deg(p) < 2021\}$;
- [ ] (c) $\{p \in \mathbb{C}[x] : p(2021i) = 0\}$.

9. Matrix $A \in \mathbb{R}^{2,2}$ is similar to matrix $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$. It follows that

- [ ] (a) A has two different real eigenvalues;
- [ ] (b) A is similar to some diagonal matrix;
- [ ] (c) A is nonsingular.

10. The set $\{a, b\}^*$ of all finite words over the alphabet $\{a, b\}$

- [ ] (a) has uncountably many countable subsets;
- [ ] (b) has uncountably many subsets that are not context-free languages;
- [ ] (c) has the same cardinality as the set of all rational numbers.

11. The lexicographic order on finite words over the alphabet $\{a, b\}$, where $a < b$, is

- [ ] (a) dense;
- [ ] (b) total;
- [ ] (c) well-founded.

12. For any equivalence relations r_1, r_2 on a nonempty set A there exists an equivalence relation on A which

- [ ] (a) includes $r_1 \cup r_2$;
- [ ] (b) equals $r_1 \cap r_2$;
- [ ] (c) is included in $r_1 - r_2$.

13. The generating function $A(x)$ of the sequence $\langle n2^n \rangle_{n \geq 0}$ satisfies

- [ ] (a) $A(0) = 0$;
- [ ] (b) $A(x) = x \cdot \frac{d}{dx}\left(\frac{1}{1-2x}\right)$;
- [ ] (c) $A(x) = \frac{1}{1-x} \cdot \frac{1}{1-2x}$.

14. Consider a $2n \times 2n$ chessboard, where $n > 0$. The number of placements of k nonattacking rooks on the main diagonals (i.e. consisting of $2n$ squares each) equals

- [ ] (a) $4n(4n-3)$ for $k = 2$;
- [ ] (b) $n2^{n+1}$ for $k = 2n - 1$;
- [ ] (c) 2^n for $k = 2n$.

15. Let G be a planar graph on 100 vertices. It follows that

- [ ] (a) the arithmetic mean of the degrees of vertices in G is ≤ 5;
- [ ] (b) the number of edges in G is ≤ 300;
- [ ] (c) the size of the largest independent set in G is ≥ 25.

16. The following relationship holds for any positive integers a, b, n, m

- [ ] (a) $a \equiv b \pmod{n}$ and $a \equiv b \pmod{m}$ implies $a \equiv b \pmod{nm}$;
- [ ] (b) $a^2 \equiv 1 \pmod{n^2}$ implies $a \equiv \pm 1 \pmod{n}$;
- [ ] (c) $am \equiv bm \pmod{nm}$ implies $a \equiv b \pmod{n}$.

17. Consider two random variables X and Y such that $E|X| < \infty$ and $E|Y| < \infty$, and random variable $Z = \max(X, Y)$. It follows that

- [ ] (a) $EZ = \max(EX, EY)$;
- [ ] (b) $EZ \geq EX$;
- [ ] (c) $E(Z \mid X \geq Y) = EX$.

18. Consider a homogeneous Markov chain with states $1, 2, 3$ and the transition matrix

$$\begin{pmatrix} \frac{1}{3} & \frac{2}{3} & 0 \\ 0 & \frac{1}{3} & \frac{2}{3} \\ \frac{2}{3} & 0 & \frac{1}{3} \end{pmatrix}$$

It follows that

- [ ] (a) the chain is irreducible;
- [ ] (b) the chain is aperiodic;
- [ ] (c) the probability that starting in state 1 after two steps the chain is in state 2 equals $\frac{4}{9}$.

19. Consider random variable $Y = e^X$, where X is a Poisson random variable with parameter λ. It follows that

- [ ] (a) $EY = e^{\lambda(e-1)}$;
- [ ] (b) $\mathrm{Var}(Y) = e^{\lambda(e^2-1)}$;
- [ ] (c) random variables X and Y are independent.

20. Two standard symmetrical, 6-sided dice, a red one and a blue one, are rolled. Let A be the event of getting 6 on the blue dice, B the event of getting 1 on the red dice, and C the event of getting the sum of values on both dice divisible by 3. It follows that

- [ ] (a) B is more probable than A;
- [ ] (b) the events A, B, C are pairwise independent;
- [ ] (c) the events A, B, C are independent.

21. Let F_X be the distribution function of a random variable X such that $EX = 10$. It follows that

- [ ] (a) if $F_X(-10) = 0$ then $P(X \geq 30) \leq \frac{1}{2}$;
- [ ] (b) if $F_X(-10) = 0$ then $P(X \geq 30) \leq \frac{1}{3}$;
- [ ] (c) if $\mathrm{Var}(X) = 1$ then $P(X \geq 30) \leq \frac{1}{5}$.

22. Tom has a deck of cards consisting of four cards with numbers 1, 1, 3, 5. He draws one of these cards at random; let X be the number on the drawn card. Then Tom rolls a standard symmetrical 6-sided dice; let K be the number on the dice. It follows that

- [ ] (a) $P(X = 5 \mid K \leq X) = \frac{1}{2}$;
- [ ] (b) $P(X = 1 \mid K \leq X) = \frac{1}{10}$;
- [ ] (c) $P(K = 6 \mid K \leq X) = \frac{1}{6}$.

23. The function $f(x) = \sin(4x)$ is interpolated on the interval $[-1, 1]$ by a Lagrange polynomial p at $n + 1$ nodes. It follows that

- [ ] (a) for $n = 2$ there is a set of nodes for which $\|f - p\|_\infty = 1$;
- [ ] (b) there exists a set of $n + 1$ nodes for which the degree of the polynomial p is at least $n + 1$;
- [ ] (c) there exists a set of 6 nodes for which $\|f - p\|_\infty \leq \frac{1}{5}$.

24. Let $b \in \mathbb{R}^m$ with $\|b\|_2 = 1$. Let A be an $m \times n$ real matrix with $m \geq n$ and $\|A\|_2 = 1$. Let $x^* \in \mathbb{R}^n$ minimise the value of the expression $\|b - Ax\|_2^2$, i.e.

$$\|b - Ax^*\|_2^2 \leq \|b - Ax\|_2^2 \qquad \text{for all } x \in \mathbb{R}^n.$$

- [ ] (a) If $n = 1$, then $x^* = \|b\|_2^2$.
- [ ] (b) If $m = n = 1$, then $\|b - Ax^*\|_2 = 0$.
- [ ] (c) If $A = I - bb^T$, then x^* is uniquely determined.

25. Consider the following sorting algorithm:

```
for (i = 0; i < n; i++){
  v = a[i]; j = i;
  while ((j > 0) && (a[j-1] > v)) {
     a[j] = a[j-1]; j--;
  }
  a[j] = v;
}
```

The worst case number of comparisons `a[j-1] > v` performed in the algorithm is

- [ ] (a) $o(n^2)$;
- [ ] (b) $\Omega(n \log n)$;
- [ ] (c) $T(n)$, where $T(1) = 0$ and $T(n) = n - 1 + T(n-1)$ for $n > 1$.

26. The amortized cost of the insert operation is $O(\log n)$ for n-element

- [ ] (a) binary search trees;
- [ ] (b) splay trees;
- [ ] (c) AVL-trees.

27. Let $\Sigma = \{a, b\}$. The following problem is decidable

- [ ] (a) whether a non-deterministic pushdown automaton $\mathcal{A}$ accepts a given word w;
- [ ] (b) whether a non-deterministic pushdown automaton $\mathcal{A}$ accepts at least one word;
- [ ] (c) whether a non-deterministic pushdown automaton $\mathcal{A}$ accepts all words over the alphabet Σ.

28. The complement of the following language (over the alphabet $\{a, b, c, d\}$) is a context-free language

- [ ] (a) $\{a^n b^n c^n d^n : n \geq 0\}$;
- [ ] (b) $\{wcw : w \in \{a, b\}^*\}$;
- [ ] (c) $\{a^p : p \text{ is a prime number}\}$.

29. The following solution of the mutual exclusion problem for three processes

```
semaphore S1 = 2;
semaphore S2 = 1;

process A[i:1..3] {
    for (;;) {
        local_section();
        P(S1);
        P(S2);
        critical_section();
        V(S2);
        V(S1);
    }
}
```

- [ ] (a) has the liveness property provided that S1 and S2 are classical Dijkstra's semaphores;
- [ ] (b) has the liveness property provided that S1 and S2 are weak semaphores;
- [ ] (c) has the safety property provided that S1 and S2 are strong semaphores.

30. Let `c` be a conditional variable declared in a monitor with Hoare semantics. Operation `signal(c)`

- [ ] (a) may block the calling proccess;
- [ ] (b) may wake a proccess awaiting on `c`;
- [ ] (c) may have no effect.