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Warsaw University
Faculty of Mathematics, Informatics and Mechanics

Exam for 2nd cycle studies of MACHINE LEARNING

20th September 2021

Solving time: 150 minutes

In each of the 30 problems there are three variants: (a), (b), and (c). For each variant you should answer if it is true, writing **`YES`** *or* **`NO`** *in the box close to it. In case of error you should cross out the box and write the correct word on its left side.*

Example of a correctly solved problem

4. Every integer of form $10^n - 1$, where n is integer and positive,

$\boxed{\texttt{YES}}$ (a) is divisible by 9;
$\boxed{\texttt{NO}}$ (b) is prime;
$\boxed{\texttt{YES}}$ (c) is odd.

You can write only in the indicated places and only the words **`YES`** *and* **`NO`***. Use pen.*

Scoring

You get "big" points (0 – 30) and "small" points (0 – 90):

- *one "big" point for each problem, in which you correctly solved all three variants;*
- *one "small" point for each correctly solved variant. So 3 "small" points in a single problem give one "big" point.*

The final result of the exam is the number

$$W = D + m/100$$

where D is the number of "big" points, and m is the number of "small" points, e.g. score 5.50 means that a candidate correctly solved 50 variants in the whole test, but gave correct answers to all three variants in a set for some five problems.

"Big" points are more important. "Small" points are just to increase resolution in case when many candidates get the same number of "big" points.

Good luck!

1. Functions $f : (0;\, 3) \longrightarrow \mathbb{R}$ i $g : [0;\, 3] \longrightarrow \mathbb{R}$ are continuous and

$$f(1) = g(1) = -7, \qquad f(2) = g(2) = 7\ .$$

It follows that

- [ ] (a) both functions are bounded;
- [ ] (b) 0 belongs to the image of both functions;
- [ ] (c) both functions are uniformly continuous.

2. Consider series $\displaystyle\sum_{n=1}^{+\infty} \frac{x^n}{\left(1+\frac{1}{n}\right)^{(n^2)}}$. It follows that

- [ ] (a) this series is absolutely convergent for each $x \in \mathbb{R}$;
- [ ] (b) for each $x \in (-2; 2)$ this series is convergent to some number $f(x)$ and so defined function $f : (-2; 2) \to \mathbb{R}$ is differentiable;
- [ ] (c) this series is convergent for $x = -e$.

3. Function $w : [0; 1] \to \mathbb{R}$ is continous, convex, and $w(0) = 0$, $w(1) = 1$. Let $I = \int_0^1 w(t)\, dt$. It follows that

- [ ] (a) $I \leq \frac{1}{2}$;
- [ ] (b) $I < \frac{1}{2}$;
- [ ] (c) $I \geq 0$.

4. The infinite sequence $\langle a_n \rangle$ of positive real numbers is monotonic and bounded. It follows that

- [ ] (a) the sequence $\langle \sin(a_n) \rangle$ is monotonic;
- [ ] (b) the sequence $\langle \cos(a_n) \rangle$ is convergent;
- [ ] (c) the sequence $\langle \log(a_n) \rangle$ is bounded.

5. A square $n \times n$ matrix A contains exactly n 1s and its remaining elements are 0s. It follows that

- [ ] (a) the determinant $\det(A) \in \{-1, 0, 1\}$;
- [ ] (b) if A is nonsingular then it is a permutation matrix;
- [ ] (c) if we exchange any 1 in A by 0 then the determinant of the resulting matrix will be zero.

6. Vectors $\vec{x}_1$, $\vec{x}_2$, $\vec{x}_3$ form a base of the linear space $\mathbb{R}^3$. It follows that

- [ ] (a) vectors $\vec{x}_1$, $\vec{x}_1 + \vec{x}_2$, $\vec{x}_1 + \vec{x}_2 + \vec{x}_3$ also form a base of $\mathbb{R}^3$;
- [ ] (b) if A is a real matrix with 3 rows and 3 columns then vectors $A\vec{x}_1, A\vec{x}_2, A\vec{x}_3$ are linearly independent;
- [ ] (c) if B is a real matrix with 6 rows and 3 columns and the $\mathrm{rank}(B) = 3$ then vectors $B\vec{x}_1$, $B\vec{x}_2$, $B\vec{x}_3$ are linearly independent.

7. Consider euclidean space $\mathbb{R}^3$ with scalar product $(\vec{x}, \vec{y}) = x_1y_1 + x_2y_2 + x_3y_3$. We are given vectors $\vec{u} = [1, 1, 1]^T$, $\vec{v} = [1, -1, 0]^T$, $\vec{w} = [1, 1, 0]^T$. It follows that

   - [ ] (a) vectors $\vec{u}$ i $\vec{v}$ are perpendicular;
   - [ ] (b) system $\vec{u}$, $\vec{v}$, $\vec{w}$ is an orthogonal base of $\mathbb{R}^3$;
   - [ ] (c) if $\vec{x}$ is the orthogonal projection of vector $\vec{w}$ onto the subspace spanned by vectors $\vec{u}$ and $\vec{v}$ then vectors $\vec{x}$ and $\vec{v}$ are perpendicular.

8. The following set is a group with operation of multiplication of complex numbers

   - [ ] (a) $\{z \in \mathbb{C} : z^{2021} = 1\}$;
   - [ ] (b) $\{z \in \mathbb{C} : |z| = 1\}$;
   - [ ] (c) $\{z \in \mathbb{C} : z - \overline{z} = 0\}$.

9. X is a linear space over the field $\mathbb{R}$ with basis $x_1, x_2, \ldots, x_n$ $(n < \infty)$ and $f : X \to X$ is a linear map. It follows that

   - [ ] (a) the system of vectors $f(x_1), f(x_2), \ldots, f(x_n)$ is also a basis of X;
   - [ ] (b) if the system $f(x_1), f(x_2), \ldots, f(x_n)$ is linearly independent then the map f is injective;
   - [ ] (c) if $f(X) = X$ then the map f is injective.

10. Let f be a function from set A to set B. It follows that

   - [ ] (a) the image of A under f is B;
   - [ ] (b) the inverse image of B under f is A;
   - [ ] (c) if X is a nonempty subset of B then the inverse image of X under f is nonempty.

11. In RGB standard, colors are denoted by triples of integers from the interval $[0; 255]$. We define an equivalence relation in the set of colors: colors $\langle r_1, g_1, b_1\rangle$ and $\langle r_2, g_2, b_2\rangle$ are *similar* if and only if $r_1 + g_1 + b_1 = r_2 + g_2 + b_2$. It follows that

   - [ ] (a) the similarity relation has more than 512 equivalence classes;
   - [ ] (b) each equivalence class of the similarity relation contains the same number of colors;
   - [ ] (c) the similarity relation is the kernel of the function f defined by $f(\langle r, g, b\rangle) = r + g + b$.

12. Consider intervals $A_n = [\frac{1}{n}; n]$ for $n \in I$ where $I = \mathbb{N} - \{0, 1\}$. It follows that

   - [ ] (a) $\left(\bigcup_{n\in I} A_n\right) \cap (-\infty; 17)$ is an open interval;
   - [ ] (b) $2 \in \bigcap_{n\in I} A_n$;
   - [ ] (c) $f \in \prod_{n\in I} A_n$ where $f(n) = n - 1$ for $n \in I$.

13. For any positive integers a and b it holds

   - [ ] (a) $a^3 \mid b^2 \quad\Rightarrow\quad a \mid b$;
   - [ ] (b) $a^2 \mid b^3 \quad\Rightarrow\quad a \mid b$;
   - [ ] (c) $a^2 \mid b^2 \quad\Rightarrow\quad a^3 \mid b^3$.

14. A connected graph G with 100 vertices is nonplanar. It follows that G

- [ ] (a) has at least 104 edges;
- [ ] (b) contains a subgraph $K_{3,3}$ or K_5;
- [ ] (c) has a vertex of degree at least 6.

15. The number of placements of n distinguishable balls in k distinguishable boxes, where n and k are positive integers, is

- [ ] (a) n^k;
- [ ] (b) $\binom{n}{k}$;
- [ ] (c) equal to the number of solutions of the equation $x_1 + x_2 + \cdots + x_k = n$ in nonnegative integers $x_1, x_2, \ldots, x_k$.

16. The sequence $\langle a_n \rangle$ is defined by the formula $a_n = |\{\langle A, x \rangle : A \subseteq \{1, \ldots, n\}, x \in A\}|$ where $|X|$ is the size of set X. It follows that

- [ ] (a) $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \infty$;
- [ ] (b) a_n is even for $n \geq 2$;
- [ ] (c) $a_n = n \cdot n!$ for $n \geq 0$.

17. We want to start a fire but we have just three matches. The probability of succesfully starting a fire with a single match is 0.4, with two joined matches – 0.6, with three joined matches – 0.8. In order to maximize the probability of starting a fire, one should

- [ ] (a) use all three matches at once;
- [ ] (b) first use two matches and then a single match;
- [ ] (c) use single matches.

18. A, B, C are events with positive probabilities. It follows that

- [ ] (a) if $P(A|B) > P(A)$ then $P(B|A) > P(B)$;
- [ ] (b) if $P(A|C) > P(A)$ and $P(B|C) > P(B)$ then $P(A|B) > P(A)$;
- [ ] (c) $P(A|C) = P(A|B)P(B|C)$.

19. Let X be a random variable with Poisson distribution with parameter λ. It follows that

- [ ] (a) $X + X$ has a Poisson distribution;
- [ ] (b) $P(X < 1) = e^{-\lambda}$;
- [ ] (c) the distribution of X is continous.

20. A Markov chain with transition matrix $\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$

- [ ] (a) has a stationary distribution;
- [ ] (b) fulfills the assumptions of the Ergodic Theorem;
- [ ] (c) has transient states.

21. A random bit generator produces bits 0 and 1. We know that each generated bit is equal to 1 with probability 1/2, but we do not know whether the generated bits are independent. Let X be a random variable equal to the number of generated 1s among 100 bits. It follows that

- [ ] (a) the expected value of X is equal to 50;
- [ ] (b) the expected value of X is equal to 50 if bits are generated independently;
- [ ] (c) the variance of X is equal to 25.

22. The execution time of a randomized algorithm is a random variable. Let μ be its expected value and σ its standard deviation. It follows that the probability that the execution time of the algorithm

- [ ] (a) exceeds 2μ is at most $1/4$;
- [ ] (b) exceeds $\mu + 2\sigma$ is at most $1/4$;
- [ ] (c) exceeds 100μ is at most $1/10^{10}$.

23. Polynomial $w(x) = 1 - x^2 + x(x-1)(x+1)^9$ interpolates the following function in points $-1, 0, 1$:

- [ ] (a) $\sin(\pi x)$;
- [ ] (b) $\cos(\frac{1}{2}\pi x)$;
- [ ] (c) $\cos(2\pi x)$.

24. The trapezoidal rule applied to function $f : [0; 1] \to \mathbb{R}$ on interval $[0; 1]$ makes it possible to compute exactly (if we ignore rounding errors) the integral $\int_0^1 f(x)\,dx$ for

- [ ] (a) $f(x) = 3 - x$;
- [ ] (b) $f(x) = x^2$;
- [ ] (c) $f(x) = \sin(x)$.

25. Let $s, t \in \{a, b\}^*$ be words of length m and n respectively, $0 < m \leq n$. In the Knuth-Morris-Pratt algorithm searching for the pattern s in the text t,

- [ ] (a) all occurrences of s in t are found in time $O(n)$;
- [ ] (b) preprocessing takes time $\Omega(n)$;
- [ ] (c) each symbol from t is compared with at most one symbol from s.

26. Let n be an integer greater than 2 and let $G = (V, E)$ be a (vertex) biconnected undirected graph with n vertices. It follows that

- [ ] (a) the height (i.e. the number of *edges* on the longest path from the root to a leaf) of each DFS-tree of G is at least 2;
- [ ] (b) if the height of some DFS-tree of G is $n - 1$ then G is a complete graph;
- [ ] (c) the root of each DFS-tree of G has at most one child.

27. If L is a regular language over alphabet Σ then the following language is regular

- [ ] (a) $\{v \in \Sigma^* : v \text{ is a prefix of some word from } L\}$;
- [ ] (b) $\{v \in \Sigma^* : v \text{ is a subsequence of some word from } L\}$;
- [ ] (c) $\{v^2 : v \in L\}$.

28. It can be decided in polynomial time whether given deterministic finite automaton

- [ ] (a) accepts all words over the input alphabet;
- [ ] (b) accepts infinitely many words;
- [ ] (c) is a minimal deterministic automaton.

29. The following mutual exclusion algorithm

```
bool want1 = false;
bool want2 = false;

process P1() {
  while (true) {
    local_section();
    want1 = true;
    while (want2) { }
    critical_section();
    want1 = false;
  }
}

process P2() {
  while (true) {
     local_section();
     want2 = true;
     while (want1) { }
     critical_section();
     want2 = false;
  }
}
```

- [ ] (a) uses busy waiting scheme;
- [ ] (b) satisfies safety property;
- [ ] (c) satisfies liveness property.

30. Let S be a strongly fair semaphore initialized to value $N > 1$. The following code is executed concurrently by $N + 1$ processes.

```
process P(int weight) {
  while (true) {
    for (i=0;i<weight;i++) P(S);
    using_resources();
    for (i=0;i<weight;i++) V(S);
    local_section();
  }
}
```

The value of parameter `weight` (in short: weight) of any process is an integer between 1 and N. It follows that

- [ ] (a) at any moment of time, function `using_resources()` is being executed by at most N processes;
- [ ] (b) at any moment of time, the sum of weights of all processes executing function `using_resources()` is at most N;
- [ ] (c) every process eventually executes `local_section()`.